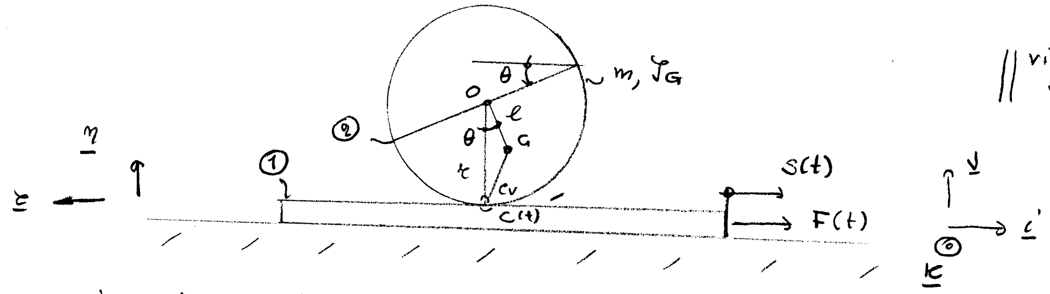
Esercitazione di martedì 6 marzo 2012



|| vincolo di rotolamento senza strisciamento ||

cinematica e dinamica

Prima osservazione : differenza fra c_v (del moto di ② risp. ad ①) che è punto ∈ ② e c(t), centro di istantanea rotazione, che è punto geometrico di contatto fra ① e ②.

Proprietà / differenze fra c_v e c(t) (nel moto relativo di ② su ①)

$c_v \in$ ② , $c(t) \notin$ ②

$\underline{v}_{cv} = \underline{0}$ $\qquad \underline{v}_c = r\dot\theta\,\underline{t}$

$\underline{a}_{cv} = r\dot\theta^2\underline{n}$ $\qquad \underline{a}_c = r\ddot\theta\,\underline{t}$

Scriviamo invece adesso le velocità assolute di c_v e c, considerando anche il fatto che il blocco ① si muove sul telaio.

Introduco osservatore $\Sigma_1 \in$ ① e scrivo (teo. comp. moti relativi)

c_v
$$\begin{cases} \underline{v}_{cv} = \underbrace{\underline{v}_{cv}^{(r)}}_{\underline{0}} + \underline{v}_{cv}^{(tr)} = \underline{v}_{cv\in ①} = \dot s\,\underline{i}\,; \\ \underline{a}_{cv} = \underline{a}_{cv}^{(r)} + \underline{a}_{cv}^{(tr)} + \underbrace{\underline{a}_{cv}^{(co)}}_{\underline{0}} = r\dot\theta^2\underline{n} + \ddot s\,\underline{i}\,; \end{cases}$$

Σ_1 <u>non ruota</u>!

C
$$\begin{cases} \underline{v}_c = \underline{v}_c^{(r)} + \underline{v}_c^{(tr)} = r\dot\theta\,\underline{t} + \dot s\,\underline{i} = (\dot s - r\dot\theta)\,\underline{i} \qquad (\underline{t} = -\underline{i}) \\ \underline{a}_c = \underline{a}_c^{(r)} + \underline{a}_c^{(tr)} = r\ddot\theta\,\underline{t} + \ddot s\,\underline{i} = (\ddot s - r\ddot\theta)\,\underline{i} \end{cases}$$

G
$$\begin{cases} \underline{v}_G = \underline{v}_{cv} + \underline{v}_{Gcv} = \dot s\,\underline{i} + \dot\theta\,\underline{k}\times \underline{c_vG} \\ \text{dato che } \underline{c_vG} = \ell\sin\theta\,\underline{i} + (r-\ell\cos\theta)\,\underline{j} \\ \underline{v}_G = (\dot s - \dot\theta(r-\ell\cos\theta))\,\underline{i} + \ell\dot\theta\sin\theta\,\underline{j} \end{cases}$$

(n.b. queste da f.f. della cinematica

G
$$\begin{cases} \underline{a}_G = \underbrace{\underline{a}_{cv}}_{r\dot\theta^2\underline{n} + \ddot s\,\underline{i}} + \underbrace{\ddot\theta\,\underline{k}\times\underline{c_vG}}_{\ell\ddot\theta\sin\theta\,\underline{j} - \ddot\theta(r-\ell\cos\theta)\,\underline{i}} \underbrace{- \dot\theta^2\,\underline{c_vG}} - \dot\theta^2(\ell\sin\theta\,\underline{i} + (r-\ell\cos\theta)\,\underline{j}) \end{cases}$$

In definitiva:

$$\underline{a}_G = r\dot\theta^2 \underline{j} + \ddot{s}\,\underline{i} + \ell\,\ddot\theta \sin\theta\,\underline{j} - \ddot\theta(r-\ell\cos\theta)\underline{j} - \dot\theta^2 \ell \sin\theta\,\underline{i} - \dot\theta^2(r-\ell\cos\theta)\underline{j}$$

$$= (\ddot{s} - \ell\dot\theta^2\sin\theta)\,\underline{i} + (\cancel{r\dot\theta^2} + \ell\ddot\theta\sin\theta - r\ddot\theta + \ell\ddot\theta\cos\theta - \cancel{r\dot\theta^2} + \ell\dot\theta^2\cos\theta)\,\underline{j}$$

Dato che $s(t)$ è imposta mi interessa scrivere eq. del moto per determinare $\theta(t)$ e quindi per evitare che compaiano forze incognite nelle eq.ni di equilibrio, posso scrivere eq. a momento del semicerchio attorno a C_V o $C(t)$, tanto come posizioni coincidono.

ossia mom. f. ext. rispetto a C_V o C è

$$\underline{M}_{CV} = \underline{M}_C = -mg\ell\sin\theta\,\underline{k} \qquad \text{(lato mom. delle forze applicate)}$$

Adesso quello che devo scrivere è il mom. f. inerzia rispetto allo stesso polo

$$\underline{M}_{CV} = \underbrace{\sum (P_i - C_V)\times m_i\,\underline{a}_{P_i}}_{M.F.I|_{CV}} \;; \qquad \underline{M}_C = \underbrace{\sum (P_i - C)\times m_i\,\underline{a}_{P_i}}_{M.F.I.|_C}$$

Adesso ricaviamo le espressioni delle M.F.I. a partire dalle def.ni dei mom ang. assoluti e relativi

a. $\underline{K}_C = \sum (P_i - C)\times m_i\,\underline{v}_{P_i}$ b. $\underline{K}_C^{(r)} = \sum (P_i - C)\times m_i(\underline{v}_{P_i} - \underline{v}_C)$

c. $\underline{K}_{CV} = \sum (P_i - C_V)\times m_i\,\underline{v}_{P_i}$ d. $\underline{K}_{CV}^{(r)} = \sum (P_i - C_V)\times m_i(\underline{v}_{P_i} - \underline{v}_{CV})$

deriviamo a.

$$\dot{\underline{K}}_C = \sum(\underline{v}_{P_i} - \underline{v}_C)\times m_i\,\underline{v}_{P_i} + \overbrace{\sum \underline{CP_i}\times m_i\,\underline{a}_{P_i}}^{M.F.I.|_C} = -\underline{v}_C \times m\,\underline{v}_G + M.F.I.|_C$$

ossia $MFI|_C = \dot{\underline{K}}_C + \underline{v}_C \times m\,\underline{v}_G$

calcoliamolo nel nostro caso:

•) $\underline{v}_C = (\dot{s} - r\dot\theta)\,\underline{i}$; $\underline{v}_G = (\dot{s} - \dot\theta(r-\ell\cos\theta))\,\underline{i} + \ell\dot\theta\sin\theta\,\underline{j}$

•) $\underline{K}_C = \sum \underline{CP_i}\times m_i\,\underline{v}_{P_i} = \sum \underline{CP_i}\times m_i(\underline{v}_{CV} + \underline{v}_{P_iCV}) =$

$= \sum \underline{CP_i}\times m_i(\dot{s}\,\underline{i} + \dot\theta\,\underline{k}\times \underline{C_VP_i}) =$ $\qquad \underline{a}\times(\underline{b}\times\underline{c}) = (\underline{a}\cdot\underline{c})\underline{b} - (\underline{a}\cdot\underline{b})\underline{c}$

$= m\,\underline{CG}\times\dot{s}\,\underline{i} + \sum m_i\,\underline{CP_i}\times(\dot\theta\,\underline{k}\times\underline{C_VP_i})$ $\qquad$ n.b. come posizione $C_V \equiv C(\bar{t})$

$= m\,\underline{CG}\times\dot{s}\,\underline{i} + \underbrace{\left(\sum m_i \|\underline{CP_i}\|^2\right)}_{J_C(\theta)}\dot\theta\,\underline{k}$

•) $\dot{\underline{K}}_C = m\left\{\dfrac{d\,\underline{CG}}{dt}\times\dot{s}\,\underline{i} + \underline{CG}\times\ddot{s}\,\underline{i}\right\} + \left\{\dfrac{dJ_C(\theta)}{d\theta}\dot\theta^2 + J_C(\theta)\ddot\theta\right\}\underline{k}$

dato che $\underline{CG} = l \sin\theta\, \underline{i} + (r - l\cos\theta)\, \underline{j}$

$$\frac{d\underline{CG}}{dt} = l\cos\theta\, \dot\theta\, \underline{i} + l \sin\theta\, \dot\theta\, \underline{j}$$

allora

$$\dot{\underline{K}}_c = -m l \sin\theta\, \dot s\, \dot\theta\, \underline{k} - m \ddot s\, (r - l\cos\theta)\, \underline{k} + \frac{dJ_c(\theta)}{d\theta}\, \dot\theta^2\, \underline{k} + J_c(\theta)\, \ddot\theta\, \underline{k}$$

adesso

$$MFI|_c = \dot{\underline{K}}_c + \underline{v}_c \times m\, \underline{v}_G = -m l \sin\theta\, \dot s\, \dot\theta\, \underline{k} - m\ddot s\,(r - l\cos\theta)\, \underline{k} + \frac{dJ_c(\theta)}{d\theta}\, \dot\theta^2\, \underline{k}$$

$$+ J_c(\theta)\, \ddot\theta\, \underline{k} + m l \sin\theta\, \dot s\, \dot\theta\, \underline{k} - m r l \sin\theta\, \dot\theta^2\, \underline{k} =$$

$$= J_c(\theta)\, \ddot\theta\, \underline{k} + m r l \sin\theta\, \dot\theta^2\, \underline{k} - m\ddot s\,(r - l\cos\theta)\, \underline{k}$$

Allora imponendo:

$$MFE|_c = MFI|_c \Rightarrow J_c(\theta)\, \ddot\theta + m r l \sin\theta\, \dot\theta^2 - m\ddot s\,(r - l\cos\theta) + m g l \sin\theta =$$

Adesso, se voglio usare il CV, conviene ricavare $MFI|_{CV}$ a partire da $\underline{K}_{CV}^{(r)}$

$$\underline{K}_{CV}^{(r)} = \sum (P_i - C_V) \times m_i\, (\underline{v}_{P_i} - \underline{v}_{CV})$$

$$\dot{\underline{K}}_{CV}^{(r)} = \underbrace{\sum (\underline{v}_{P_i} - \underline{v}_{CV}) \times m_i\, (\underline{v}_{P_i} - \underline{v}_{CV})}_{= \underline{0}} + \sum \underline{C_V P_i} \times m_i\, (\underline{a}_{P_i} - \underline{a}_{CV}) =$$

$$= \underbrace{\sum \underline{C_V P_i} \times m_i\, \underline{a}_{P_i}}_{MFI|_{CV}} - m\, \underline{C_V G} \times \underline{a}_{CV}$$

da cui

$$MFI|_{CV} = \dot{\underline{K}}_{CV}^{(r)} + m\, \underline{C_V G} \times \underline{a}_{CV}$$

- $\underline{C_V G} = l \sin\theta\, \underline{i} + (r - l\cos\theta)\, \underline{j}$
- $\underline{a}_{CV} = r\dot\theta^2\, \underline{j} + \ddot s\, \underline{i}$
- $\underline{K}_{CV}^{(r)} = \sum \underline{C_V P_i} \times m_i\, (\dot\theta\, \underline{k} \times \underline{C_V P_i}) = \left(\sum m_i \|\underline{C_V P_i}\|^2\right) \dot\theta\, \underline{k} =: J_{CV}\, \dot\theta\, \underline{k} = J_c(\bar\theta)\, \dot\theta\, \underline{k}$

cioè questo J_{CV} è costante perché $CV \in (2)$

dunque quando lo derivo

$$\dot{\underline{K}}_{CV}^{(r)} = J_c(\bar\theta)\, \ddot\theta\, \underline{k}$$

- altri termini $\underline{C_V G} \times m\, \underline{a}_{CV} = m\left[l \sin\theta\, \underline{i} + (r - l\cos\theta)\, \underline{j}\right] \times \left[\ddot s\, \underline{i} + r\dot\theta^2\, \underline{j}\right]$

$$= m r l \sin\theta\, \dot\theta^2\, \underline{k} - m\ddot s\,(r - l\cos\theta)\, \underline{k}$$

Allora rimettendo tutto insieme

$$MFI\big|_{cv} = \left\{ \mathcal{I}_c(\theta)\,\ddot\theta + m r \ell \sin\theta\,\dot\theta^2 - m\ddot s\,(r-\ell\cos\theta)\right\}\underline{k}$$

Allora imponendo:

$$MFE\big|_{cv} = MFI\big|_{cv} \Rightarrow \mathcal{I}_c(\theta)\,\ddot\theta + m r\ell\sin\theta\,\dot\theta^2 - m\ddot s\,(r-\ell\cos\theta) + mg\ell\sin\theta = 0$$

ovviamente l'eq.ne è la stessa trovata prima.

Quanto vale $\mathcal{I}_c(\theta)$?

$$\mathcal{I}_c(\theta) = \mathcal{I}_G + m\,\|\underline{GC}(\theta)\|^2 \quad e$$

$$\|\underline{GC}\|^2 = \underline{GC}\cdot\underline{GC} = \underline{CG}\cdot\underline{CG} = [\ell\sin\theta\,\underline{i} + (r-\ell\cos\theta)\underline{j}][\ell\sin\theta\,\underline{i} + (r-\ell\cos\theta)\underline{j}] =$$

$$= \ell^2\sin^2\theta + (r^2 + \ell^2\cos^2\theta - 2r\ell\cos\theta) =$$

$$= \ell^2 + r^2 - 2r\ell\cos\theta \quad // \text{ anche teo. di } \underline{\text{Carnot}}$$

allora

$$\mathcal{I}_c(\theta) = \mathcal{I}_G + m(\ell^2 + r^2 - 2r\ell\cos\theta)$$

In alternativa avrei potuto usare la ben nota:

$$\underline{M}_A = \dot{\underline{K}}_G^{(r)} + \underline{AG}\times m\,\underline{a}_G$$

scegliendo come $A \to cv$ o $A \to c$ tanto in termini di posizione (questo compare come info nella formula) sono equivalenti

avrei scritto

$$\underline{M}_{cv} = \dot{\underline{K}}_G^{(r)} + \underline{cvG}\times m\,\underline{a}_G$$

$$\circ)\ \dot{\underline{K}}_G^{(r)} = \mathcal{I}_G\,\ddot\theta\,\underline{k} \quad ; \quad \underline{cvG} = \ell\sin\theta\,\underline{i} + (r-\ell\cos\theta)\,\underline{j}$$

$$\underline{a}_G = (\ddot s - \ell\dot\theta^2\sin\theta)\,\underline{i} + (\ell\ddot\theta\sin\theta - r\ddot\theta + \ell\ddot\theta\cos\theta + \ell\dot\theta^2\cos\theta)\,\underline{j}$$

$$m\{\underline{cvG}\times\underline{a}_G\} = m\underline{k}\{\ell^2\sin^2\theta\,\ddot\theta - r\ell\sin\theta\,\ddot\theta + \ell^2\sin\theta\cos\theta\,\ddot\theta + \ell^2\dot\theta^2\sin\theta\cos\theta +$$

$$\underbrace{-(r-\ell\cos\theta)(\ddot s - \ell\dot\theta^2\sin\theta)\}}$$

$$-r\ddot s + r\ell\dot\theta^2\sin\theta + \ell\ddot s\cos\theta - \ell^2\dot\theta^2\sin\theta\cos\theta$$

viene identico espandendo.

Con eq.ni di Lagrange avrei scritto

$$T = T^{(1)} + T^{(2)}$$

$$T^{(1)} = \frac{1}{2} M \dot{s}^2$$

con punto A arbitrario

$$T^{(2)} = \text{da König generalizzato} = \frac{1}{2} m v_A^2 + \underline{v}_A \cdot (m \underline{v}_{GA}) + \frac{1}{2} J_A \dot{\theta}^2$$

n.b. se $A \equiv G$ teo. König classico

caso $A \to C_V$

$$T^{(2)} = \frac{1}{2} m \|\underline{v}_{cv}\|^2 + \frac{1}{2} J_{cv}(\theta) \dot{\theta}^2 + \underline{v}_{cv} \cdot (m \underline{v}_{Gcv})$$

$$\underline{v}_{cv} = \dot{s}\, \underline{i} \quad ; \quad J_{cv}(\theta) = \left[J_G + m(r^2 + \ell^2 - 2r\ell\cos\theta) \right]$$

Allora

$$T^{(2)} = \frac{1}{2} m \dot{s}^2 + \frac{1}{2} J_c(\theta) \dot{\theta}^2 - m \dot{s} \dot{\theta} (r - \ell\cos\theta)$$

Globalmente:

$$T = \frac{1}{2}(M+m) \dot{s}^2 + \frac{1}{2} J_c(\theta) \dot{\theta}^2 - m \dot{s} \dot{\theta} (r - \ell\cos\theta)$$

Con $U(\theta=0) = 0 \longrightarrow U(\theta) = mg\ell(1-\cos\theta)$

allora g.d.l. sono s e θ

Eq.ni di Lagrange

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} + \frac{\partial U}{\partial \theta} = Q_\theta = 0$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{s}}\right) - \frac{\partial T}{\partial s} + \frac{\partial U}{\partial s} = Q_s$$

in generale $$Q_{q_h} = \sum_{i=1}^{m_f} \underline{F}_i \cdot \frac{\partial(\underline{OP}_i)}{\partial q_h} + \sum_{j=1}^{m_M} \underline{M}_j \cdot \frac{\partial(\underline{\varepsilon}_j)}{\partial q_h}$$

$$Q_s = F(t)\,\underline{i} \cdot \frac{d(s\,\underline{i})}{ds} = F(t)\,\underline{i} \cdot \underline{i} = F(t)$$

$$\frac{\partial T}{\partial \dot{\theta}} = J_c(\theta)\dot{\theta} - m\dot{s}(r - \ell\cos\theta)$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) = \frac{dJ_c(\theta)}{d\theta}\dot{\theta}^2 + J_c(\theta)\ddot{\theta} - m\ddot{s}(r - \ell\cos\theta) - m\dot{s}\dot{\theta}\ell\sin\theta$$

$$\frac{\partial T}{\partial \theta} = \frac{1}{2}\dot{\theta}^2 \frac{dJ_c(\theta)}{d\theta} - m\ell\dot{s}\dot{\theta}\sin\theta \quad ; \quad \frac{\partial U}{\partial \theta} = mgl\sin\theta$$

Allora se vado a considerare $J_C(\theta) = J_G + m\left[r^2 + \ell^2 - 2r\ell\cos\theta\right]$
e faccio tutti i conti ottengo:

$$\left[J_G + m(r^2 + \ell^2 - 2r\ell\cos\theta)\right]\ddot{\theta} + mr\ell\sin\theta\,\dot{\theta}^2 - m\ddot{s}\,(r - \ell\cos\theta) + mg\ell\sin\theta = 0.$$

La 2ª viene

$$(M+m)\ddot{s} - m(r - \ell\cos\theta)\ddot{\theta} - m\ell\sin\theta\,\dot{\theta}^2 = F(t)$$ $\longleftrightarrow$ eq. algebrica di din inversa che consente il calcolo di $F(t)$

per farla discendere da <u>Newton</u> occorre scrivere l'equilibrio alla traslazione orizzontale del sistema completo 1 + 2 così le forze interne non vi compaiono.