Fun with stacking blocks

John F. Hall
California Institute of Technology, Pasadena, California 91125

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How can a given number of rigid, rectangular blocks be stacked in a planar arrangement to produce the maximum overhang over a support edge? To answer this question, three cases of increasing complexity are considered: single-wide stacks, multiwide stacks that do not rely on friction, and multiwide stacks that do rely on friction. The solution to the first case has existed for more than 150 years; the answer to the second case is attempted in this paper; and the considerable complexity of the third is demonstrated. Many mathematical aspects of block stacking are discussed, and a new challenge is posed. The analysis uses the principles of static equilibrium and stability and free body diagrams, key concepts in classical mechanics.
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I. INTRODUCTION

Textbooks on engineering mechanics contain many simple, yet interesting, applications of the principles of static equilibrium. One particular problem involving a stack of rigid, rectangular blocks is especially intriguing. Its answer is not intuitive, even in the ideal world in which the problem is formulated.

The problem is to place N blocks in an overhanging stack (Fig. 1) and calculate the overhang D. Block 1 (the top block) is positioned so that its balance point is on the upper right corner of block 2; blocks 1 and 2 together are positioned so that their balance point is on the upper right corner of block 3, etc., until finally the entire stack of N blocks is positioned so that its balance point is on the corner of the support. Thus, block 1 is on the verge of tipping about the upper right corner of block 2; blocks 1 and 2 together are on the verge of tipping about the upper right corner of block 3, etc. The block positions can be determined by considering a series of free bodies of the individual blocks beginning with block 1 and satisfying the equilibrium of forces and moments. Free bodies are constructed with the realization that forces acting between adjacent blocks and at the support are vertical forces concentrated at the balance points.

Local overhangs are determined one by one as the free bodies are analyzed: $d_1=\frac{1}{2}b$, $d_2=\frac{1}{4}b$, $d_3=\frac{1}{6}b$, $\cdots d_N=(1/2N)b$, where b is the block length, and

$$d_i=\frac{b}{2i} \quad (i=1 \text{ to } N) \tag{1}$$

is the overhang of block i relative to block $i+1$ directly below (or the support for $i=N$). The total overhang D is calculated from the sum

$$D=\sum_{i=1}^{N} d_i=\frac{b}{2}\cdot\sum_{i=1}^{N}\frac{1}{i}. \tag{2}$$

Because Eq. (2) is a harmonic series, D approaches infinity as the number of blocks N approaches infinity. This result is unintuitive.

The block stacking problem has been exposed to a wide audience, and the items being stacked are variously referred to as blocks, books, bricks, slabs, cards and coins. It is used to demonstrate the harmonic series in mathematics,[1,2] and has been noted as a curiosity in the physics literature.[3,4] There is even an interactive Web site[5] where one can stack green colored blocks, which turn red if an instability is detected. The problem has a long history in textbooks of engineering mechanics,[6–13] dating back to the 1800's at the University of Cambridge where it is cited in the context of material relevant to the Examinations for the Mathematical Tripos. Perhaps its origins are even older.

The purpose of this paper is to discuss some work related to the original block stacking problem as well as more general cases. An example is given to demonstrate that by using some blocks as counterweights, greater overhangs than given by Eq. (2) can be achieved for the same number of blocks. Then an attempt is made to answer the question, given N blocks, what stacking arrangement produces the largest overhang? As far as can be determined, this question has not been formerly considered; although the overhang given by Eq. (2) has been incorrectly claimed to be the maximum achievable.[2,3] Following the treatment of the maximum overhang problem, a new block stacking challenge is posed. This paper is intended to be an enjoyable exploration of a stimulating class of problems that involves only simple principles of statics.

II. ASSUMPTIONS FOR BLOCK STACKING; SINGLE-WIDE STACKS

Several assumptions are made to facilitate the analysis. Each block is rigid, rectangular, the same size, the same density, and has its center of gravity at the centroid; the supporting surface is rigid and horizontal; the gravity field is vertical and uniform. Each block can be positioned exactly as desired and no disturbing influences are present. These conditions establish the ideal world of block stacking. In addition, each block is to be laid flat with its long axis horizontal. Only planar stacks will be considered.

The stacking method described in Sec. I is consistent with these assumptions. It employs a single-wide stack, that is, consisting of a single block on the support with the remaining blocks stacked above one-on-one. In this case, the only forces acting are vertical, including the block weights and the forces between the blocks and at the support. Thus, the only parameter on which the overhang depends is the number of blocks N and the block length b. In particular, the block height and any coefficient of friction are not involved in single-wide stacks.

Although it may seem obvious that the method of stacking described in Sec. I would achieve the maximum overhang

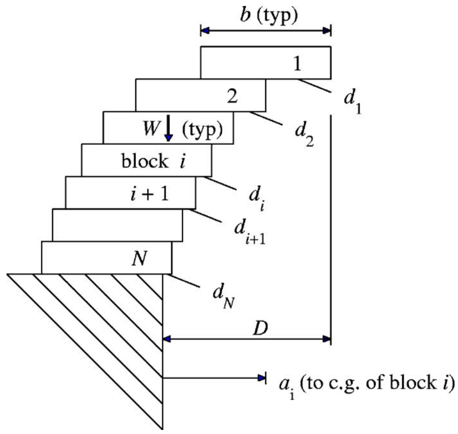

Fig. 1. Stack of N blocks overhanging the support a distance D, showing local overhangs d_i and distances a_i from the support edge to a block's center of gravity.

for N blocks in a single-wide stack, a proof is desirable. Refer to the single-wide stack shown in Fig. 1, where a_i is the horizontal distance between the edge of the support and the center of gravity of block i. Moment equilibrium requires that the following equations be satisfied:

$$\frac{1}{i}(a_1 + a_2 + \cdots + a_i) = a_{i+1} + \frac{b}{2} - \Delta_i \quad (i = 1 \text{ to } N-1), \tag{3a}$$

$$\frac{1}{N}(a_1 + a_2 + \cdots + a_N) = -\Delta_N, \tag{3b}$$

where the Δ_i are arbitrary non-negative distances. Equation (3) states that the center of gravity of a group of i blocks at the top of the stack ($i=1$ to $N-1$) must not lie beyond the right edge of block $i+1$, or for the entire stack ($i=N$), its center of gravity must not lie beyond the support edge. These N equations can be solved for a_1 by eliminating a_2 through a_N:

$$a_1 = C \cdot b - \Delta_N - \sum_{i=1}^{N-1} \frac{1}{i+1} \cdot \Delta_i, \tag{4}$$

where C is a positive constant. Because the Δ_i are non-negative, a_1 is maximized when all of the Δ_i are zero. Thus, the Δ_i in Eq. (3) can be set to zero, and then Eq. (3) is modified by a series of substitutions to produce:

$$a_i = a_{i+1} + \frac{b}{2i} \quad (i = 1 \text{ to } N-1), \tag{5a}$$

$$a_N = \frac{b}{2N} - \frac{b}{2}. \tag{5b}$$

From Eq. (5), the local overhangs are found to be

$$d_i = a_i - a_{i+1} = \frac{b}{2i} \quad (i = 1 \text{ to } N-1), \tag{6a}$$

$$d_N = a_N + \frac{b}{2} = \frac{b}{2N}, \tag{6b}$$

which are the same as given in Sec. I. This result completes the proof for a single-wide stack.

One possible flaw in this analysis is the assumption that the top block is the one that overhangs the most. In fact, for any single-wide stack of at least two blocks, block 1 can be shifted a distance $\frac{1}{2}b$ to the left while block 2 is shifted a distance $\frac{1}{2}b$ to the right, putting the second block in the maximum overhang position while maintaining equilibrium. The top block acts as a counterweight in this case. The maximum overhang is unchanged, but the configuration that achieves the maximum overhang is now seen to be nonunique. This case is believed to be the only way that the top block does not set the maximum overhang for single-wide stacks.

A good approximation to Eq. (2) is

$$D = 0.2886 \cdot b + \frac{b}{2} \cdot \ln(N + 0.5), \tag{7}$$

whose inverse relation is

$$N = \exp\left(\frac{2D}{b} - 0.5772\right) - 0.5. \tag{8}$$

From Eq. (8) the multiplicative factor by which the number of blocks has to be increased to extend the overhang a full block length b can be calculated by taking the ratio of $N(D+b)$ to $N(D)$. This ratio is a measure of the efficiency of a stacking arrangement. As D/b increases, the ratio quickly approaches a limit that is termed the stacking factor S. From Eq. (8),

$$S = e^2 \approx 7.39. \tag{9}$$

Thus, approximately 7.39 times as many blocks are required to produce an overhang of length, say, $D=5b$ as compared to $D=4b$.

This relatively high value of S is one reason why achieving a substantial overhang is difficult if one were to stack actual blocks. Therefore, the computer is the preferred block stacking tool. Figure 2 shows a computer generated plot of a stack for $N=40$, with local overhangs computed according to Eq. (1). In this case, $D=2.139b$.

III. NOT-SO-PERFECT CONDITIONS FOR BLOCK STACKING

The question addressed in this section is how many of the ideal world conditions listed in Sec. II are actually necessary for the unintuitive result of Eq. (2) to hold, that is, $D\rightarrow\infty$ as $N\rightarrow\infty$. Theoretically, the infinite overhang is still possible by using blocks of different length and density, and centers of gravity not necessarily at the centroid. The method of stacking would be similar to that described in Sec. I where successive groups of blocks are positioned with their balance point on the upper right corner of the next block (or support) below. Also, a support surface that is not exactly horizontal and top and bottom block surfaces that are flat but not exactly parallel (one way in which a block could be nonrectangular) could be compensated for by stacking the blocks in a predetermined order, dependent on their shapes, to keep the current stacking surface essentially horizontal. So, these de-

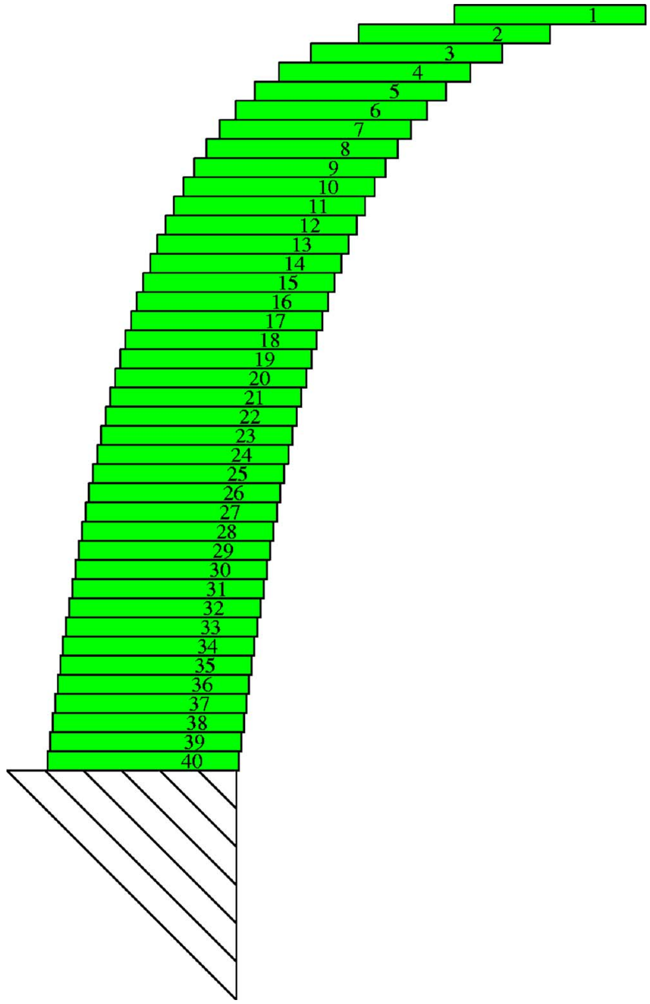

Fig. 2. A single-wide stack for $N=40$ achieving an overhang $D=2.139b$. Local overhangs are determined according to Eq. (1).

partures from ideal conditions do not seem to be limiting factors either. On the other hand, the rigid assumption of the blocks and support seems necessary for the result $D\rightarrow\infty$ as $N\rightarrow\infty$ to hold. If the blocks and support are made of a flexible material, the stack will lean forward due to strains induced from bending in the blocks and from the narrow contacts between the blocks and at the support. This effect places a limit on the overhang that could be achieved, although using lighter blocks partially compensates.

Another aspect of the assumption that the blocks are rectangular is that the corners are perfectly sharp. This assumption allows the balance point of each group of i blocks at the top of the stack to be placed exactly over the right edge of block $i+1$ directly below (or the support edge if $i=N$), ensuring that the maximum local overhang is reached. To see if this assumption can be relaxed, consider more realistic blocks whose corners are replaced by tapers beginning a horizontal distance x in from the block edge (see Fig. 3), the same x for every block and also for the support. When such blocks are stacked, the balance point for each group of blocks is shifted to where the taper begins, that is, the distance x from the right edge of the next block below, as is also shown in Fig. 3. Thus, block 1 is placed on block 2 with a local overhang $d_1=\frac{1}{2}b-x$. The other local overhangs are

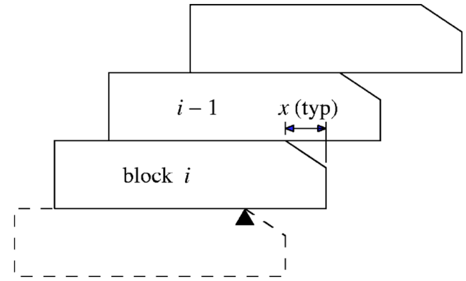

Fig. 3. Single-wide stack with tapered blocks. The triangle marks the balance point for the group of i blocks above.

computed by the same method discussed previously, considering a series of free bodies and satisfying equilibrium of forces and moments. The result is

$$d_i=\frac{b}{2i}-\frac{x}{i}, \tag{10}$$

which gives a total overhang for N blocks of

$$D=\sum_{i=1}^{N}\left(\frac{b}{2i}-\frac{x}{i}\right). \tag{11}$$

If $x<\frac{1}{2}b$, D still approaches infinity as N approaches infinity. This result is surprising because the shifts x can be much greater than the local overhangs present in the lower part of the stack. The shifting does reduce the efficiency of the stack; the stacking factor can be computed as

$$S=\exp\left(\frac{2}{1-2x/b}\right), \tag{12}$$

which varies from 7.39 at $x=0$, to 9.23 at $x=0.05b$, to 12.18 at $x=0.10b$.

Shifting balance points to the left can also be used to impart positive rotational stability to the stack[4] if the shifts are beyond the points where the tapering begins. Otherwise, the rotational stability is only neutral. A stable stack offers resistance to finite disturbances.

A final assumption is that the blocks can be placed exactly where desired. As an example of how this assumption can be assessed, suppose that the blocks are being stacked by a machine that is only able to set a finite number of local overhang distances, say, multiples of a block length fraction λb. For simplicity, take $x=0$ and assume the local overhangs are set exactly to multiples of λb. If $\lambda=0.01$, then d_1 would be set to the desired value $0.5b$, and d_2 would be set to the desired value $0.25b$. However, d_3 would have to be set to $0.16b$ instead of the desired value $0.1666\cdots b$. Subsequent local overhangs are calculated by satisfying force and moment equilibrium for the series of free bodies as before, but choosing the greatest λb multiple that is less than or equal to the computed overhang. Eventually, a situation will be reached where there is a sequence of zero local overhangs, a single one at λb, a longer series of zero local overhangs, another one at λb, etc. This method of stacking also produces the result $D\rightarrow\infty$ as $N\rightarrow\infty$, with the stacking factor S given by

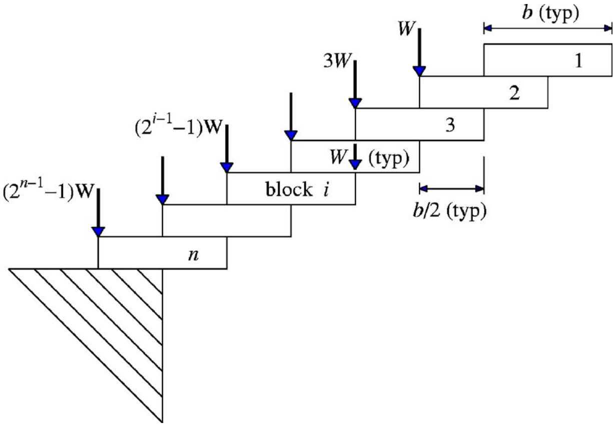

Fig. 4. Blocks on the leading edge of a multiwide stack with constant overhang $\frac{1}{2}b$, also showing minimum counterweight forces.

$$S=\left(\frac{1}{1-2\lambda}\right)^{1/\lambda}, \tag{13}$$

which approaches e^2 for small λ. For $\lambda=0.01$, $S\approx 7.54$, representing only a small decrease in stacking efficiency.

Thus there is considerable robustness in stacking blocks to achieve a large overhang. Additional variables could be examined such as uncertainty in the block positions. The remainder of the paper returns to the ideal world of block stacking as characterized by the original list of assumptions.

IV. MULTIWIDE STACKS

A multiwide stack is one that has a single block on the support and one or more blocks in each layer above. Each upper block rests on one block or two adjacent blocks on the next layer below. With this definition, single-wide stacks are included as a special case. Forces are transferred only between blocks on adjacent layers and between the base block and the support.

Multiwide stacks are able to use blocks to provide counterweight. This stacking method can increase the maximum overhang for a given number of blocks, that is, a lower stacking factor S. For example, consider a stack whose leading edge consists of n blocks placed with equal local overhangs $\frac{1}{2}b$ (see Fig. 4). Such a stack needs counterweight to be stable. By applying force and moment equilibrium to a series of free bodies, the minimum counterweight forces in terms of the block weight W can be computed; these are also shown in Fig. 4. The use of minimum counterweight imparts a neutral rotational stability to the stack.

The counterweight forces are actually applied through additional blocks stacked on those forming the leading edge. Thus, to obtain a total overhang D of $\frac{1}{2}nb$, $N=2^n-1$ blocks are required, which gives a stacking factor $S=4$, a little more than half the value of S for the single-wide stack. Figure 5(a) shows how the blocks providing the counterweight can be positioned for $n=5$, which involves $N=31$ blocks. To clarify the correspondence to the counterweight forces, the seven blocks supplying the counterweight to block 4 are shown shaded.

A symmetric version of this type of stack exists for any value of n, achieving the same overhang with the same number of blocks. An example is shown in Fig. 5(b) for $n=5$.

V. OPTIMUM STACKS OF TYPE V

The fact that the counterweighted stacking method presented in Sec. IV produces greater overhang for a given number of blocks, compared to the original single-wide stacking method, raises the question as to what is the absolute maximum overhang that can be achieved for a given value of N. This question turns out to be very difficult to answer and involves finding the optimal stacking arrangement. A multiwide stack of N blocks that reaches maximum overhang mobilizes friction forces between the blocks, and so the maximum overhang becomes a function of the coefficient of friction μ between the blocks. In addition, the block aspect ratio h/b, where h is the block height, becomes a parameter. This general case is discussed in the Appendix where results for $N=4$ and $N=5$ are presented as illustrations.

A simpler case, one in which equilibrium of the stack requires only vertical forces between the blocks to be present, is more amenable to analysis and is examined in the following. These solutions for maximum overhang apply only to perfectly smooth blocks or for blocks that are negligibly thin compared to their length. This special case will be referred to

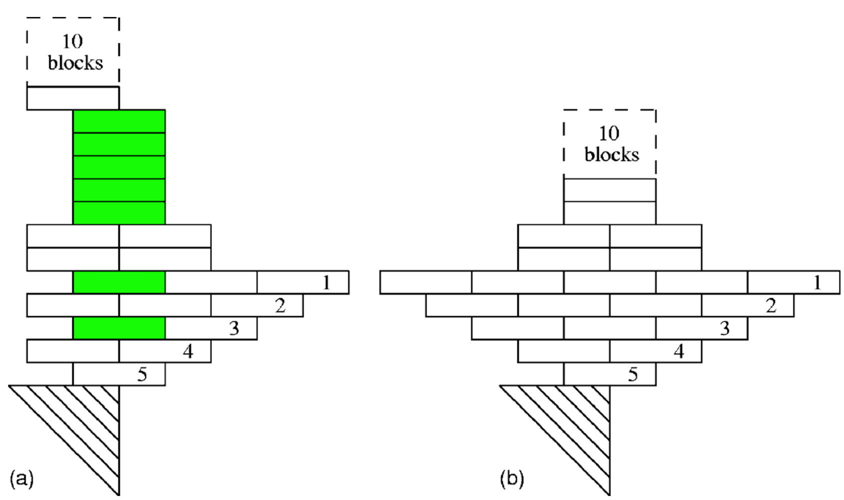

Fig. 5. Two multiwide stacks of constant local overhang $\frac{1}{2}b$; $N=31$; $n=5$; and $D=\frac{5}{2}b$. Unsymmetric [part (a)] and symmetric [part (b)] versions.

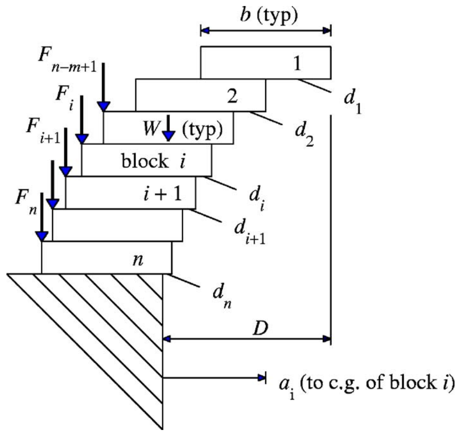

Fig. 6. Blocks on the leading edge of a multiwide stack with variable local overhangs d_i to be determined to maximize the overhang D. Counterweight forces are applied to the lowest m blocks of the leading edge.

as type V (for only vertical forces acting); the more general case is type F (with friction forces acting as well, as discussed in the Appendix).

As in Sec. IV, the stacks considered here have a leading edge of n blocks counterweighted with additional blocks (see Fig. 6). However, the positions of the blocks in the leading edge are now to be determined to maximize the overhang D of the stack, which is assumed to be set by block 1. In addition, n itself is unknown and must also be determined to maximize D. Counterweight forces are applied to all n leading blocks or only to the bottom m of these ($m \leqslant n$), with m as well needing to be determined to maximize D. The counterweight forces are denoted by $F_{n-m+1}, \cdots F_n$. These forces are computed to provide minimum counterweight, and so with these forces acting, each group of i blocks from the top on the leading edge would be on the verge of tipping. This condition implies that the forces acting between the blocks on the leading edge, R_i for the force between blocks i and $i+1$ (or R_n for the force between block n and the support), are concentrated at the upper right corner of block $i+1$ (or the support corner if $i=n$). Note that $R_i = i \cdot W$ for $i=0$ to $n-m$ and that $R_n = N \cdot W$, where W is the block weight. A free-body diagram of block i is shown in Fig. 7. All forces are vertical.

Summation of moments on the free body in Fig. 7 about any point on the left edge of block i leads to

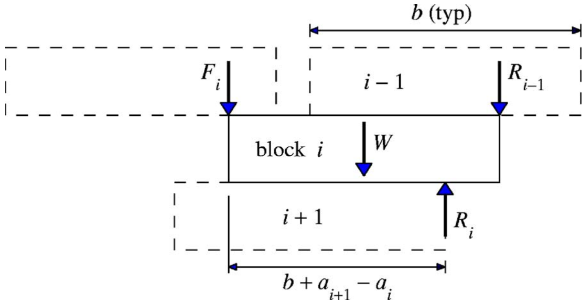

Fig. 7. Free-body diagram of block i in the leading edge of a multiwide stack.

$$(b + a_{i+1} - a_i) \cdot R_i = \frac{b}{2} w + b \cdot R_{i-1} \quad (i = 1 \text{ to } n). \tag{14}$$

From these n equations, the block positions $a_2, a_3, \cdots a_n$ can be eliminated, resulting in a single equation for a_1 in terms of N and the unknown R_i for $m \geqslant 2$:

$$a_1 = f(R_{n-m+1}, \cdots R_{n-1}, N). \tag{15}$$

To maximize a_1, set

$$\frac{\partial f}{\partial R_i} = 0 \quad (i = n - m + 1 \text{ to } n - 1), \tag{16}$$

which results in a set of equations from which the unknown R_i can be found. These relations are nonlinear but have a simple form:

$$R_i^2 = \left(\frac{W}{2} + R_{i-1} \right) \cdot R_{i+1} \quad (i = n - m + 1 \text{ to } n - 1), \tag{17}$$

and are easily solved with an iterative algorithm.

The positions of the blocks in the leading edge can now be calculated recursively from bottom to top (decreasing i) as

$$a_n = \frac{b}{2} - \frac{b}{2N} - \frac{b}{NW} \cdot R_{n-1}, \tag{18a}$$

$$a_i = a_{i+1} + b - b \cdot \frac{R_i}{R_{i+1}} \quad (i = n - 1 \text{ to } n - m + 1), \tag{18b}$$

$$a_i = a_{i+1} + \frac{b}{2i} \quad (i = n - m \text{ to } 1; \quad i \neq n), \tag{18c}$$

where Eq. (18b) is used when $m \geqslant 2$, and Eq. (18c) is used when $n > m$. The stack overhang is then

$$D = a_1 + \frac{b}{2}. \tag{19}$$

The counterweight forces F_i are not needed to find D. However, these forces are required to determine the positions of the blocks providing the counterweight. For this purpose, the F_i can be found from

$$F_1 = R_1 - W, \tag{20a}$$

$$F_i = R_i - i \cdot W - \sum_{j=1}^{i-1} F_j \quad (i = 2 \text{ to } n). \tag{20b}$$

The above procedure for computing D must be repeated for every possible combination of n and m. For example, there are 26 possible choices of n and m for $N=10$, consisting of one single-wide stack $n=10/m=0$ and 25 multiwide stacks where $n=1$ to 9 and $m=1$ to $\min(n, 10-n)$. Of the 26 combinations, $n=4/m=4$ produces the largest overhang. Table I lists the values of n and m that produce the largest value of D for $N=1$ to 40, as well as the resulting values of D. All of these stacks for $N \geqslant 3$ are counterweighted. The $N=2$ stack is a special case in that both a noncounterweighted version ($n=2/m=0$) and a counterweighted version ($n=1/m=1$) reach the same maximum overhang.

Before the selection of an n/m combination that maximizes D for a given N can be accepted, a verification must be made that a valid stack of the blocks providing the counterweight can be constructed. The existence of such a counter-

Table I. Values of n and m that maximize the overhang D of type V stacks for $N=1$ to 40.

N	n	m	D/b
1	1	0	0.500
2	1	1	0.750
2	2	0	0.750
3	2	1	1.000
4	2	2	1.168
5	3	2	1.305
6	3	3	1.437
7	3	3	1.530
8	4	4	1.632
9	4	4	1.715
10	4	4	1.787
11	5	5	1.859
12	5	5	1.925
13	5	5	1.985
14	5	5	2.038
15	6	6	2.093
16	6	6	2.144
17	6	6	2.191
18	6	6	2.235
19	7	7	2.277
20	7	7	2.319
21	7	7	2.358
22	7	7	2.395
23	7	7	2.431
24	8	8	2.465
25	8	8	2.499
26	8	8	2.531
27	8	8	2.562
28	8	8	2.592
29	9	9	2.620
30	9	9	2.649
31	9	9	2.677
32	9	9	2.703
33	9	9	2.729
34	9	9	2.754
35	10	10	2.778
36	10	10	2.802
37	10	10	2.825
38	10	10	2.848
39	10	10	2.870
40	10	10	2.892

stack is not guaranteed for all n/m combinations. For $N=10$, the $n=9/m=1$ combination cannot be realized because block 8 in the leading edge occupies too much of the top surface of the bottom block 9, onto which the single counterweight block 10 must also be placed. No block interpenetration is allowed. Also, in general, the blocks providing the counterweight must produce the correct counterweight forces [Eq. (20)]; these blocks must be in equilibrium; and all vertical forces between block layers must be compressive and directly transmitted through block-to-block contact.

Experience has shown that a valid counterstack always seems to exist for the particular n/m combination leading to the maximum overall D for a given N, although this observation has not been proven. In fact, except for $N \leqslant 3$, such counterstacks seem to be nonunique. Thus, the values of n, m and D in Table I for $N=1$ to 40 are believed to be for optimum, realizable type V stacks.

The method employed for computing a counterstack involves some trial and error; no fully automated procedure has been developed. To overcome some of the nonuniqueness, the forces between the blocks in any two adjacent layers are placed at the upper corners of the blocks in the lower layer, as if rotation is impending about these corners. Such a stack has neutral stability for two types of mechanisms. One of these is a rigid-body rotation about the corner of the support, and the other involves a nonrigid-body set of block motions that opens up all of the block-to-block contact surfaces through rotation and slipping at the aforementioned block upper corners. These two mechanisms are illustrated in Fig. 8 for a $N=4$ stack. Such a configuration is referred to as fully mechanized, and it greatly facilitates the analysis. Figure 9 shows fully mechanized, optimum type V stacks for $N=1$ to 8.

The type of nonuniqueness overcome by considering only fully mechanized stacks can be illustrated with the $N=4$ stack in Fig. 9. Blocks 3 and 4 of the counterstack can be displaced horizontally toward each other by equal amounts (thus, keeping their combined center of gravity stationary), but not so far as to slide block 3 off block 1 or to penetrate block 1 with block 4. All of these configurations satisfy equilibrium, but are not fully mechanized because blocks 3 and 4

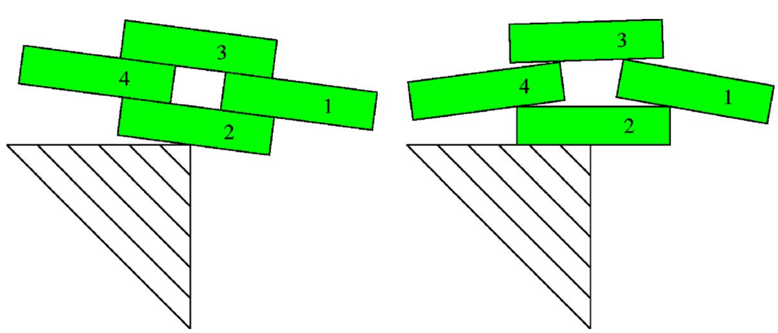

Fig. 8. Fully mechanized stack for $N=4$ showing the rigid-body rotational mechanism [part (a)] and the opening mechanism [part (b)].

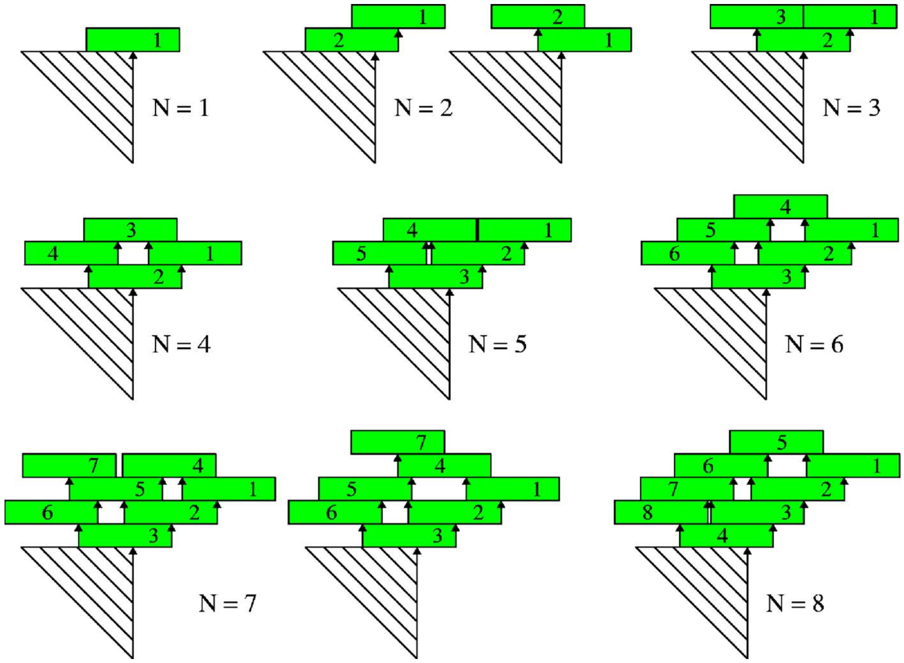

Fig. 9. Fully mechanized, type V stacks for $N=1$ to 8 that achieve maximum overhang. Triangles denote contact points where vertical forces between blocks and at the support are concentrated.

would move as a unit in the opening mechanism. This nonuniqueness seems to be present for $N \geqslant 4$.

Another type of nonuniqueness of the block positions in the counterstack involves block connectivity. Consider thetwo $N=7$ stacks in Fig. 9, for which $n=3/m=3$ produces the optimum stack. Block 7 must be placed on block 4, 5, or 6. Valid solutions can be obtained for block 7 atop either block 4 or block 5 (see Fig. 9). Except for $N=8$, such nonuniqueness seems to be present for $N \geqslant 7$. As N becomes large, many nonunique connectivities emerge. Even so, valid counterstacks become difficult to find. Figure 10 shows an optimum stack for $N=40$ that was obtained through much effort.

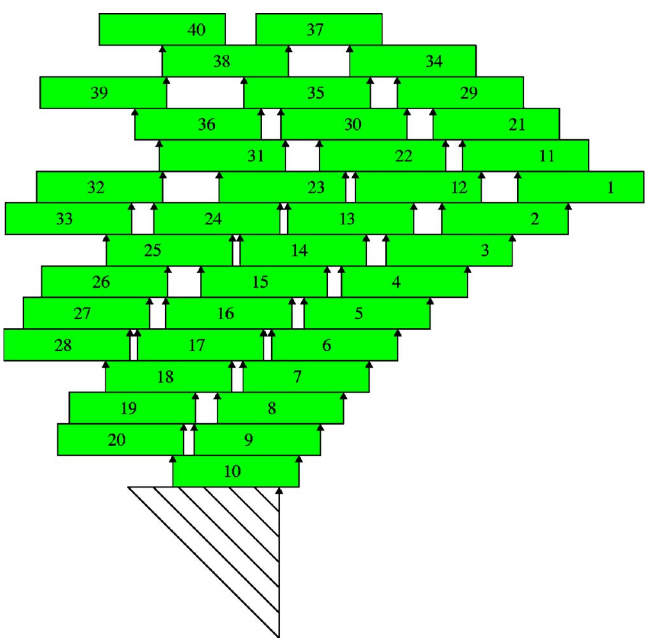

Fig. 10. Fully mechanized, type V stack for $N=40$ that achieves maximum overhang ($D=2.892b$). Triangles denote contact points where vertical forces between blocks and at the support are concentrated.

A computer calculation to determine parameters of optimum type V stacks for N up to 10 000 generated the results in Table II. Trends seen in Table I continue. Namely, for the optimum stack, m appears to equal n, and n becomes a smaller fraction of N as the stack size increases. A good fit to the data in Table II is given by

$$N = 3.013\,36 \cdot \exp\left(\frac{D}{b}\right) - 3.4722 \cdot \exp\left(\frac{D}{2b}\right) + 0.425, \tag{21}$$

where the dominant term $e^{D/b}$ is believed to be the correct functional form. This term yields the stacking factor

$$S = e \approx 2.72, \tag{22}$$

which is lower than the stacking factors given previously. Also, the data in Table II indicate that as N increases by a factor of e to extend the overhang a length b, n increases by

Table II. Values of n and m that maximize the overhang D of type V stacks for N producing overhangs of approximately integer multiples of b from b to $8b$.

N	n	m	D/b
3	2	1	1.000 000 0
14	5	5	2.038 216 9
46	11	11	3.011 459 4
140	21	21	4.004 648 2
406	37	37	5.001 531 7
1147	64	64	6.000 540 3
3190	109	109	7.000 001 3
8794	183	183	8.000 049 7

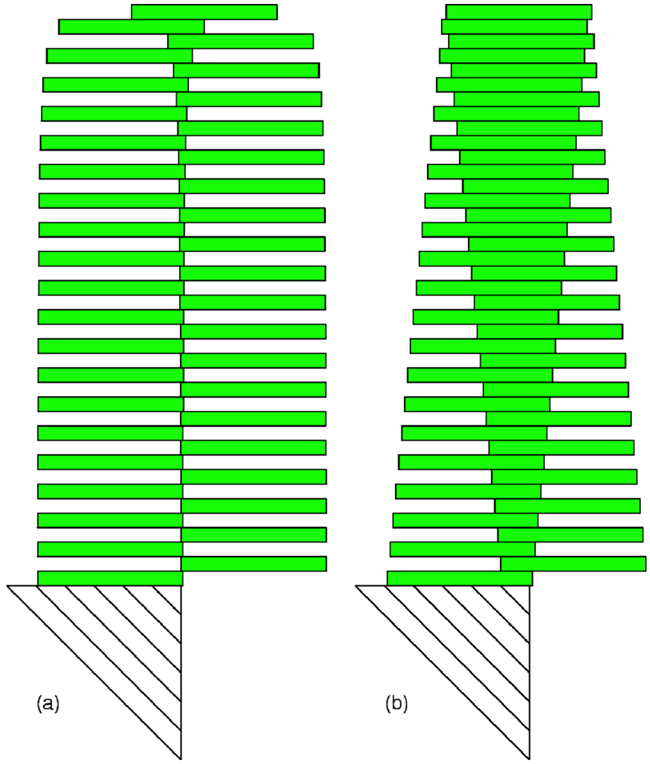

Fig. 11. Tree stacks made with $x=0$ [part (a)] and a linear variation of x [part (b)].

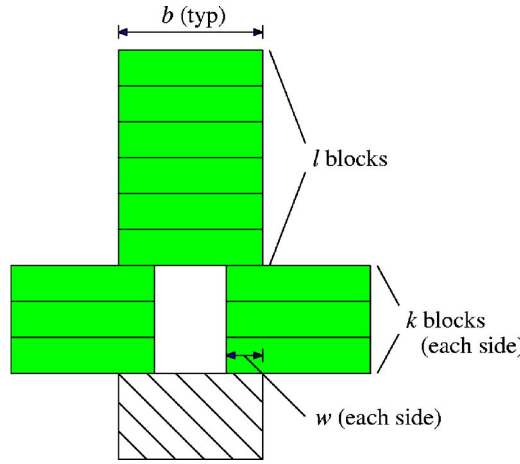

Fig. 12. Use of two vertical columns capped by another vertical column to create an enclosed opening. The length of the support is same as the block length.

a factor of $e^{1/2}$, which is consistent with n becoming a smaller fraction of N for larger stacks.

Finally, two curious features of optimum type V stacks are emphasized. First is the involvement of the constant e in the relations among N, D and n, and in the stacking factor S. This constant also appears in Eq. (9) for the stacking factor S of single-wide stacks, but there the reason for its presence is clear. For optimum multiwide stacks, there is no obvious suggestion in the mathematics that e will play a role. Perhaps the presence of e indicates that a continuum version of these optimum stacks exists for large N. Second is the fact that the procedure for computing the maximum overhang D using Eqs. (17)–(19) does not have to compute the positions of any of the blocks that provide the counterweight forces, only to assume that a valid counterstack exists (which always seems to be the case). Only the positions of the leading edge blocks need to be found, along with n and m, and a simple algorithm has been described to do this.

VI. OTHER STACKS

One property of a single-wide stack is that any group of i blocks from the top of the stack can be rotated as a group 180 deg about a vertical axis through the balance point of the group, a maneuver that will be called flipping. For example, the two $N=2$ stacks in Fig. 9 can be viewed as flipped versions of each other. For the $N=40$ stack shown in Fig. 2, if the top 39 blocks are flipped, then the top 38 blocks, etc., continuing to the top, the tree stack shown in Fig. 11(a) is produced. This stack has neutral stability. If x shifting (see the previous discussion on tapered blocks) is applied prior to flipping, a stack with positive stability is retained. An example is the stack shown in Fig. 11(b), which was produced with an initial x shift varying linearly from $0.5b$ at the top to $0.1b$ at the bottom. Some intriguing patterns can be made in this way.

Various other problems can be posed other than achieving maximum overhang. One of these is to build a stack atop a block-wide support that encloses the maximum opening per total number of blocks used. If the area of the opening is quantified by the number of block areas (one block area equals b times the block height), then this ratio, Z, is dimensionless. As the problem is posed, the opening should be continuous.

Consider the symmetric stack in Fig. 12 consisting of two overhanging vertical columns of k blocks each, capped with a vertical column of l blocks. The overhanging stacks each occupy a length w of the support. The ratio Z is

$$Z=\frac{k(1-2w/b)}{2k+l}. \tag{23}$$

A relationship between w, k and l can be found by using moment equilibrium on an appropriate free body on which only vertical forces are considered (a type V stack):

$$\frac{w}{b}=\frac{k}{2k+l}. \tag{24}$$

The substitution of w/b into Eq. (23) yields

$$Z=\frac{\beta}{(2+\beta)^2}, \tag{25}$$

where $\beta=l/k$. Maximizing Z with respect to β leads to $\beta=2$ and

$$Z=\tfrac{1}{8}=0.125. \tag{26}$$

This value of Z could be increased if friction forces were mobilized (type F stack).

The stacking type shown in Fig. 12 may or may not be the one that results in the largest value of Z. Many other types of stacks with enclosed openings are possible, such as shown in Fig. 13, which consists of two free-standing stacks. The one on the right side is of the type depicted in Fig. 2, but has been flipped at its base to make room for the vertical stack on the left. This particular stack with 49 blocks has Z

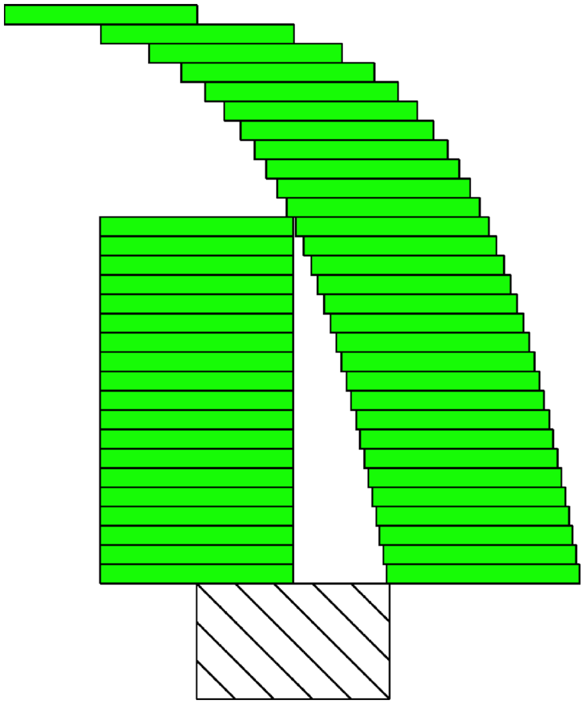

Fig. 13. Use of two free-standing stacks to create an enclosed opening. The length of the support is the same as the block length.

=0.109, which is less than Z=0.125 for the other stacking method. Finding the stacking type that produces the maximum Z is a challenging problem and is left for the interested reader.

VII. CONCLUSIONS

The major part of this paper attempts to answer the question: Given N rigid, rectangular blocks, what is the optimum stacking arrangement that maximizes the overhang over a support edge? The answer involves using most blocks in a counterstacking role. If only vertical forces are considered to act between blocks (type V stack), an algorithm can be formulated to compute the locations of the blocks on the leading edge of the optimum stack, which determines the maximum overhang. However, this calculation is subject to verification that a valid solution for the blocks providing the counterweight can be found. For all values of N examined so far, this verification step succeeded, but no proof has been offered that a valid counterstack always exists when the overhang is maximum. If the blocks are stacked to mobilize friction forces (type F stack), then greater overhang can be achieved. However, a type F stack is a much more complicated situation to analyze, and no general procedure has been developed.

Although no practical applications have been claimed for the results presented, the block stacking problem could pose a worthy test for general optimization algorithms. To reduce some of the nonuniqueness involved, the original question could be restated, for example, to maximize the minimum gap between adjacent blocks, in addition to maximizing the overhang.

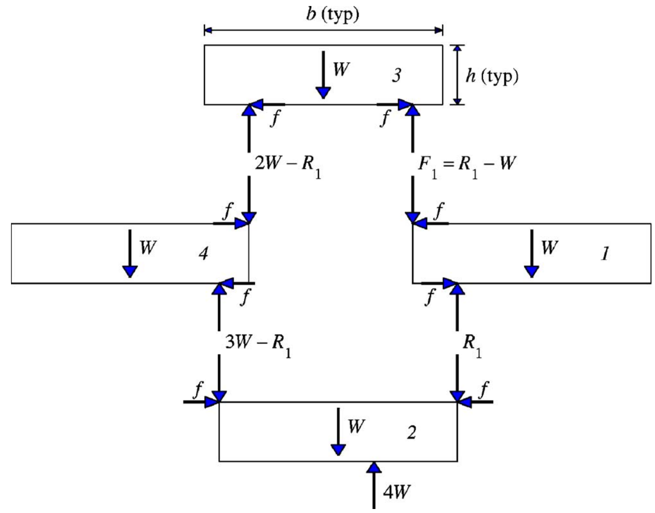

Fig. 14. Free-body diagrams of the blocks of a N=4 stack with friction forces mobilized (type F stack).

APPENDIX: OPTIMUM STACKS OF TYPE F

A type F stack is one in which friction forces are involved, as compared to a type V stack in which all forces between the blocks are vertical. All single-wide stacks are type V. Multiwide stacks can be either, and greater overhangs can be obtained with type F stacks depending on the friction coefficient μ and the block aspect ratio h/b.

Figure 14 shows free-body diagrams of the blocks in a N=4 stack including friction forces. Vertical forces between blocks have been located in the same manner as done in the analysis of type V stacks at maximum overhang. Friction forces arise naturally as the stack is spread out to increase the overhang. Consideration of the horizontal equilibrium of the blocks of the N=4 stack shows that all the friction forces are equal in magnitude. These friction forces, denoted by f, exert stabilizing couples on blocks 1 and 4 and thereby allow these blocks to be spread further apart than if only vertical forces were acting. The analysis of the N=4 stack is complicated by the presence of the extra unknown f and by the fact that f could be controlled either by impending sliding of block 3 on block 1 or by impending sliding of block 3 on block 4. Nevertheless, a solution can be found without great difficulty,

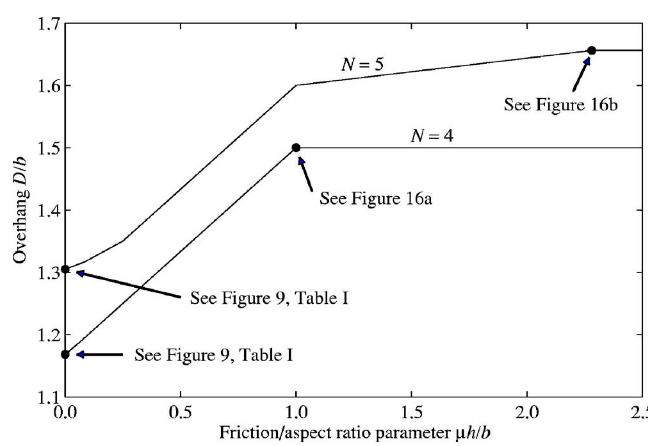

Fig. 15. Relationship between maximum overhang D and the friction/aspect ratio parameter $\mu h/b$ for N=4 and N=5 stacks of type F.

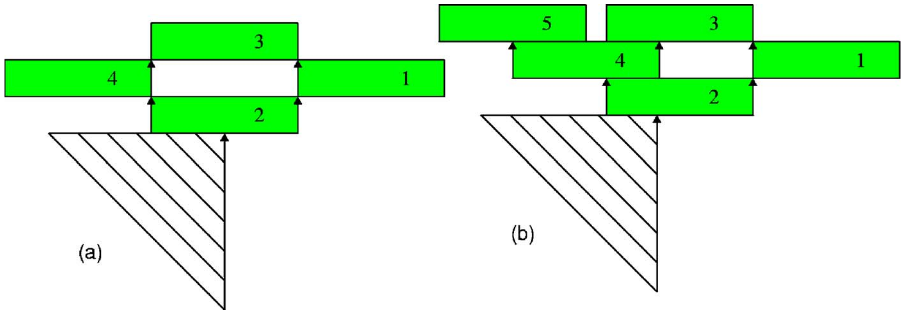

Fig. 16. Final configurations for $N=4$ [part (a)] and $N=5$ [part (b)] stacks of type F beyond which further increases in $\mu h/b$ cannot increase the maximum overhang ($D=1.5$b and $D=1.656$b, respectively). Triangles denote contact points where forces between blocks and at the support are concentrated.

and it reveals the maximum overhang D to be a function only of the combined parameter $\mu h/b$. This relation is plotted in Fig. 15. The overhang cannot be increased further once $\mu h/b$ reaches unity; this final configuration is shown in Fig. 16(a) ($D=1.5b$).

A similar solution for $N=5$ has also been completed, and again the maximum overhang D depends only on $\mu h/b$. This relation is also plotted in Fig. 15. For $\mu h/b \leq 0.25$, D is controlled by a $n=3/m=2$ configuration, and for $\mu h/b \geq 0.25$, the controlling configuration switches to $n=2/m=2$. The overhang cannot be increased further once $\mu h/b$ reaches 2.281; this final configuration is shown in Fig. 16(b) ($D=1.656b$). For higher values of N, the complexity of the analysis increases severely, and no general solution procedure has been obtained as for type V stacks.

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