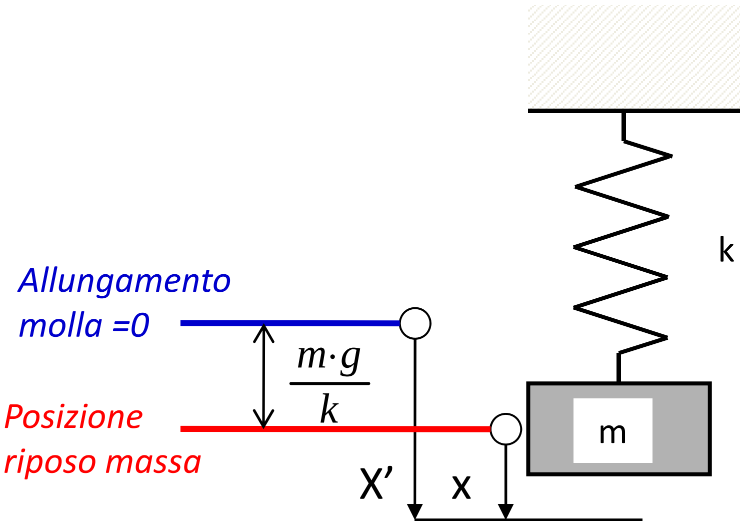
OSCILLAZIONE LIBERA SISTEMA 1 G.D.L. NON SMORZATO



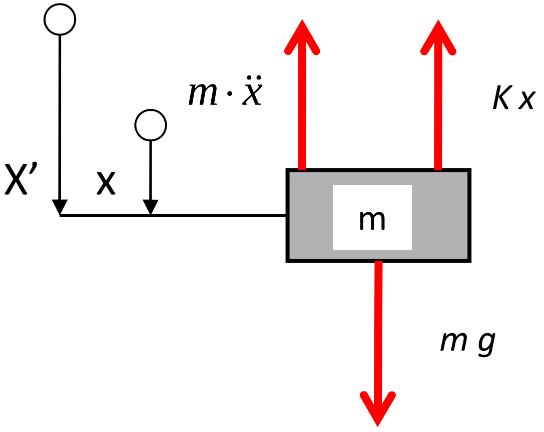
$$X' = x + \frac{mg}{k}$$

$$\ddot{X}' = \ddot{x}$$

$$-m\ddot{X}' - kX' + mg = 0$$

$$-m\ddot{x} - k\left(x + \frac{mg}{k}\right) + mg = 0$$

Analisi delle forze agenti



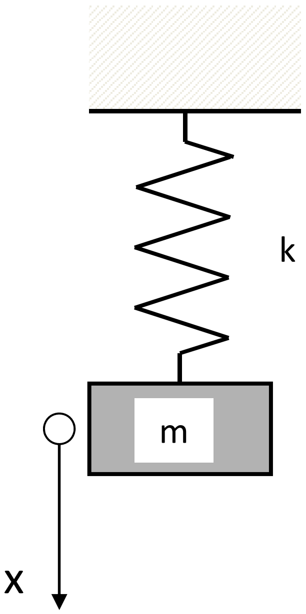
$$-m\ddot{x} - kx - mg + mg = 0$$

$$m\ddot{x} + kx = 0$$

Equazione del moto non influenzata dalla forza peso

OSCILLAZIONE LIBERA SISTEMA 1 G.D.L. NON SMORZATO

Sistema ad 1 g.d.l.



$$m\ddot{x} + kx = 0$$

$$\ddot{x} + \frac{k}{m}x = \ddot{x} + \omega_n^2 x = 0$$

$$x(t) = C_1 e^{z_1 t} + C_2 e^{z_2 t}$$

$$z^2 + \omega_n^2 = 0 \qquad z_{1,2} = \pm i\omega_n$$

$$x(t) = C_1 e^{i\omega_n t} + C_2 e^{-i\omega_n t}$$

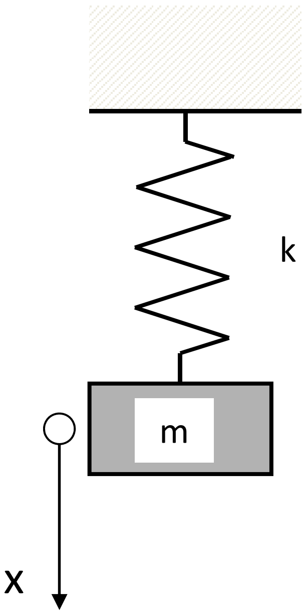
$$x(t) = C_3 \cos(\omega_n t) + C_4 \sin(\omega_n t)$$

$$x(t) = C_5 \cos(\omega_n t + \varphi_6)$$

$$x(t) = C_7 \cos(\omega_n t + \varphi_8)$$

OSCILLAZIONE LIBERA SISTEMA 1 G.D.L. NON SMORZATO

Sistema ad 1 g.d.l.



$$E_c = \frac{1}{2} m \cdot \dot{x}^2$$

$$E_p = \frac{1}{2} k \cdot x^2$$

Energia totale

$$\Rightarrow \quad E = E_c + E_p = \frac{1}{2} m \cdot \dot{x}^2 + \frac{1}{2} k \cdot x^2$$

Soluzione trovata

$$x(t) = A \sin(\omega_n t)$$

$$\dot{x}(t) = A\omega_n \cos(\omega_n t)$$

$$E = \frac{1}{2} m \cdot (A\omega_n \cos(\omega_n t))^2 + \frac{1}{2} k \cdot (A \sin(\omega_n t))^2 =$$

$$= \frac{A^2}{2}\left(m\,\omega_n^2 \cos^2(\omega_n t) + k \sin^2(\omega_n t)\right)$$

$$E = \text{cost} \;\rightarrow\; m\omega_n^2 = k$$

$$\omega_n = \sqrt{\frac{k}{m}}$$

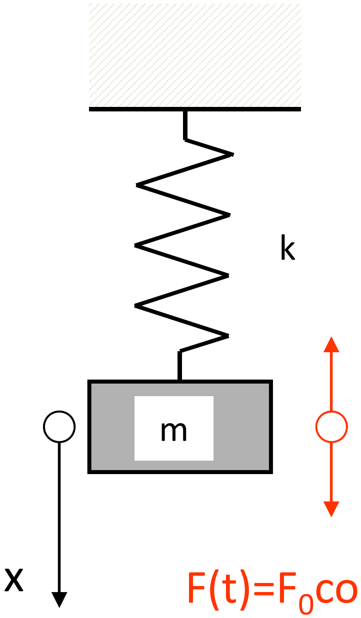
$$E = \frac{A^2}{2}\left(m \frac{k}{m} \cos^2(\omega_n t) + k \sin^2(\omega_n t)\right) =$$

$$= \frac{A^2 k}{2}\left(\cos^2(\omega_n t) + \sin^2(\omega_n t)\right) = \frac{A^2 k}{2} = \text{cost}$$

OSCILLAZIONE FORZATA SISTEMA 1 G.D.L. NON SMORZATO

Sistema ad 1 g.d.l.

$$m\ddot{x} + kx = F_0 e^{i\Omega t} = F_0 \cos(\Omega t) \qquad (\Omega \neq \omega_n)$$



F(t)=F_0cos(Ωt)

$$x(t) = \boxed{C_1 e^{z_1 t} + C_2 e^{z_2 t}} + \boxed{X e^{i\Omega t}}$$

Integrale generale omogenea associata

Integrale particolare non omogenea

Verifica validità integrale particolare non omogenea:

$$x(t) = X e^{i\Omega t} \qquad \dot{x}(t) = i\Omega X e^{i\Omega t}$$

$$\ddot{x}(t) = -\Omega^2 X e^{i\Omega t}$$

$$-m\Omega^2 X e^{i\Omega t} + kX e^{i\Omega t} = F_0 e^{i\Omega t}$$

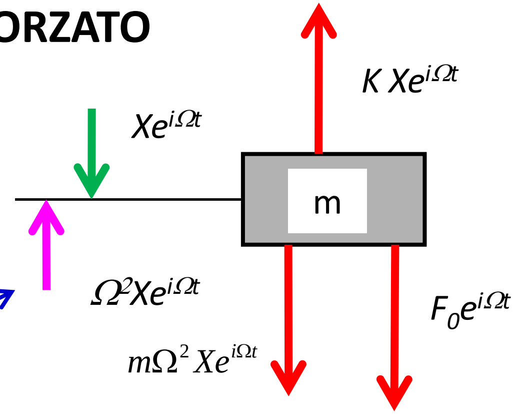
$$X = \frac{F_0}{k - m\Omega^2} = \frac{F_0}{k}\frac{1}{1 - \dfrac{m\Omega^2}{k}} = \frac{F_0}{k}\frac{1}{1 - \left(\dfrac{\Omega}{\omega_n}\right)^2}$$

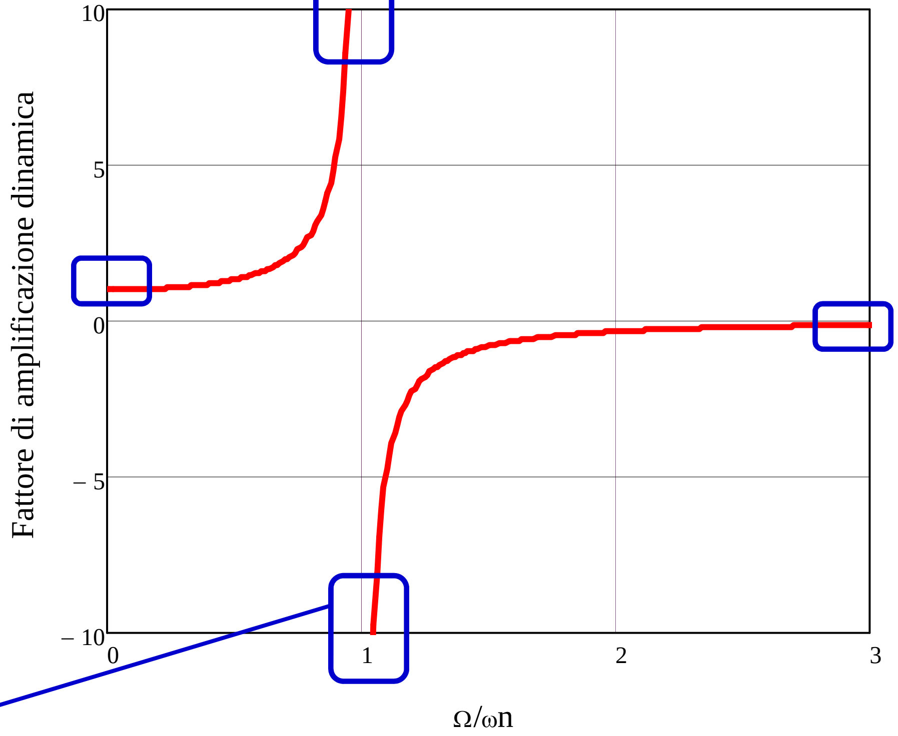
OSCILLAZIONE FORZATA SISTEMA 1 G.D.L. NON SMORZATO

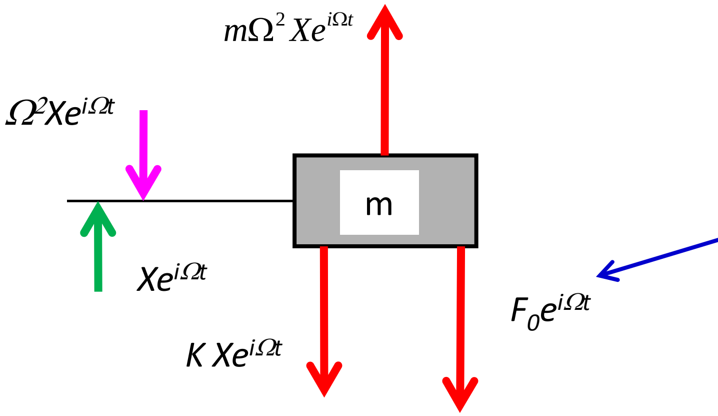
$$X = \boxed{\frac{F_0}{k}}\boxed{\frac{1}{1-\left(\frac{\Omega}{\omega_n}\right)^2}}$$

Freccia statica

Fattore di amplificazione dinamica



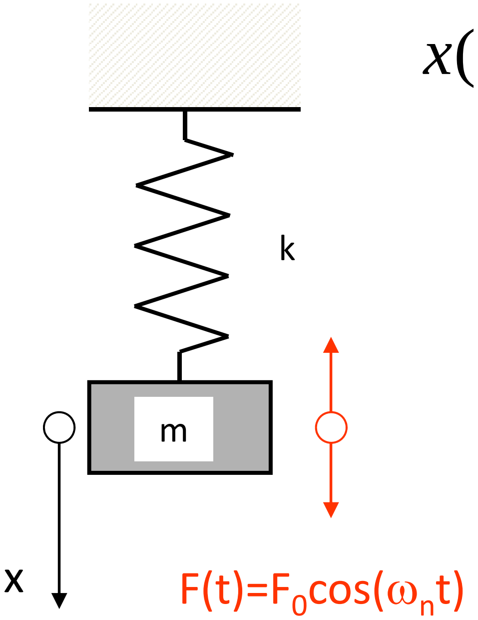




OSCILLAZIONE FORZATA IN RISONANZA SISTEMA 1 G.D.L. NON SMORZATO

Sistema ad 1 g.d.l.

$$m\ddot{x}+kx=F_0 e^{i\omega_n t} \qquad \ddot{x}+\omega_n^2 x=\frac{F_0}{m}e^{i\omega_n t}$$



$$x(t)=\boxed{Xte^{i\omega_n t}}$$

Integrale particolare
non omogenea

$$\dot{x}(t)=Xe^{i\omega_n t}\left(1+i\omega_n t\right)$$

$$\ddot{x}(t)=\omega_n Xe^{i\omega_n t}\left(2i-\omega_n t\right)$$

$$\omega_n Xe^{i\omega_n t}\left(2i-\omega_n t\right)+\omega_n^2 Xte^{i\omega_n t}=\frac{F_0}{m}e^{i\omega_n t}$$

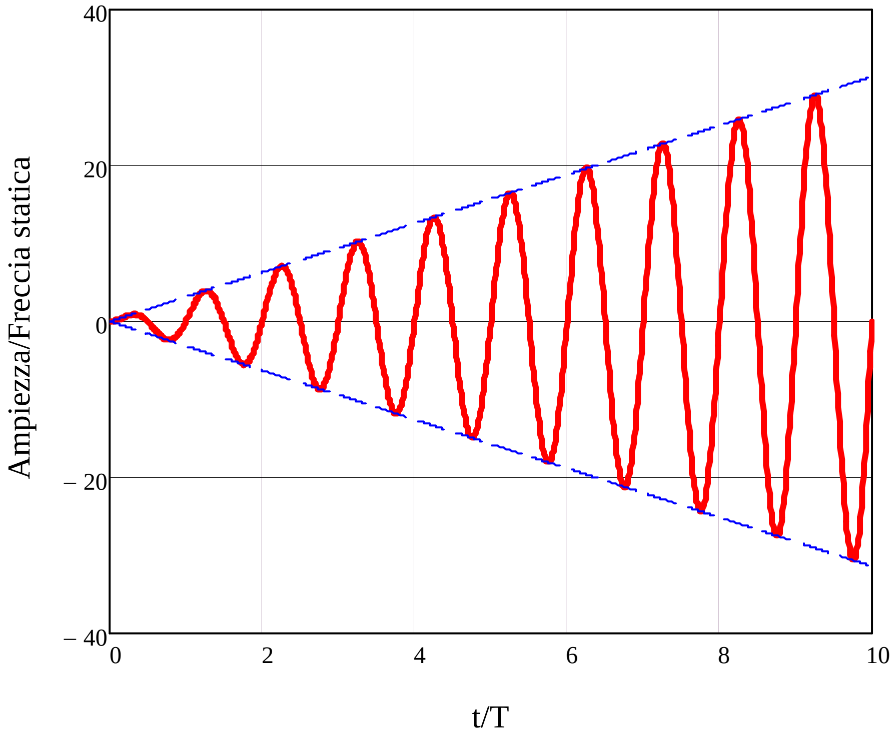
$$2i\omega_n X-\omega_n^2 Xt+\omega_n^2 Xt=\frac{F_0}{m}$$

$$X=\frac{F_0}{m}\frac{1}{2i\omega_n}=\frac{F_0}{k}\frac{\omega_n}{2i}$$

OSCILLAZIONE FORZATA IN RISONANZA SISTEMA 1 G.D.L. NON SMORZATO

$$x(t) = \frac{F_0}{k}\frac{\omega_n}{2i} t e^{i\omega_n t} = -i\pi \frac{F_0}{k}\frac{t}{T} e^{i\frac{2\pi}{T}t}$$

Dato che, in corrispondenza di ω_n, il sistema è in grado di oscillare senza cedere energia all’esterno, tutto il lavoro fatto dalla forza applicata si tasforma in aumento del suo contenuto enegetico.



OSCILLAZIONE FORZATA IN RISONANZA SISTEMA 1 G.D.L. NON SMORZATO

$$\boxed{\ddot{x}}+\boxed{\omega_n^2 x}=\boxed{\frac{F_0}{m}e^{i\omega_n t}}$$

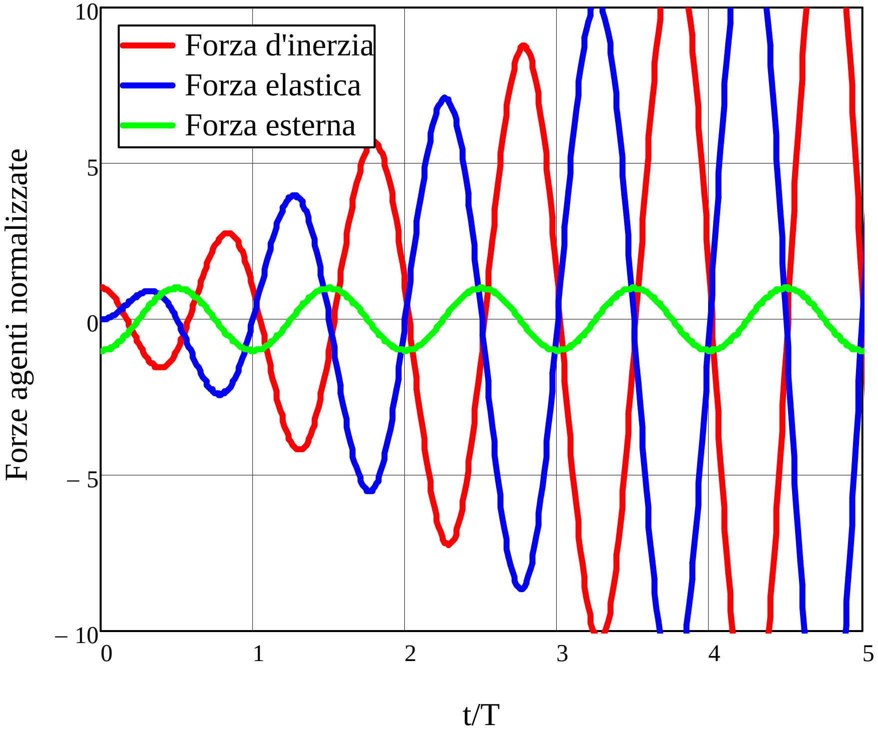
$$x(t)=-i\frac{F_0}{2k}\omega_n t e^{i\frac{2\pi}{T}t}$$

Forza d'inerzia

Forza elastica

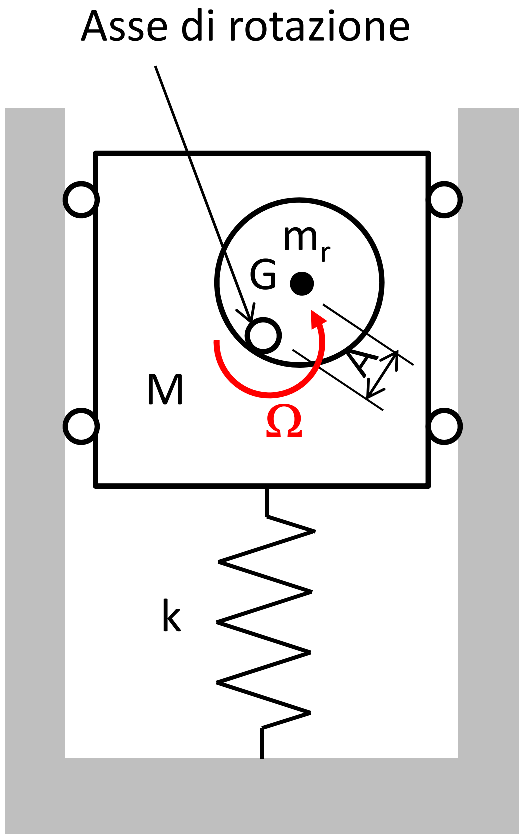
Forza esterna

$$\boxed{\frac{F_0}{k}\omega_n^2 e^{i\omega_n t}(1+i\frac{\omega_n}{2}t)}-\boxed{i\frac{F_0}{2k}\omega_n^3 t e^{i\omega_n t}}=\boxed{\frac{F_0}{k}\omega_n^2 e^{i\omega_n t}}$$



OSCILLAZIONE FORZATA SISTEMA 1 G.D.L. NON SMORZATO

Forzante armonica di ampiezza proporzionale al quadrato della pulsazione



$$M\ddot{x} + kx = m_r A \Omega^2 e^{i\Omega t}$$

$$\ddot{x} + \omega_n^2 x = \frac{m_r A}{M}\Omega^2 e^{i\Omega t}$$

$$x(t) = Xe^{i\Omega t} \qquad \dot{x}(t) = i\Omega X e^{i\Omega t}$$

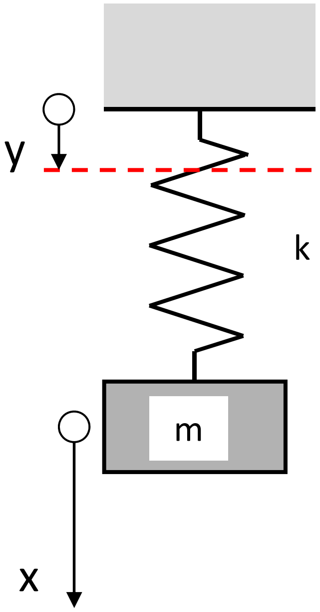
$$\ddot{x}(t) = -\Omega^2 X e^{i\Omega t}$$

$$-\Omega^2 X e^{i\Omega t} + \omega_n^2 X e^{i\Omega t} = \frac{m_r A}{M}\Omega^2 e^{i\Omega t}$$

$$X = \frac{\frac{m_r A}{M}\Omega^2}{\omega_n^2 - \Omega^2} = \frac{\frac{m_r A}{M}}{\frac{\omega_n^2}{\Omega^2} - 1}$$

OSCILLAZIONE FORZATA SISTEMA 1 G.D.L. NON SMORZATO

Eccitazione per moto del supporto



$$m\ddot{x}+k(x-y)=0$$

$$m\ddot{x}+kx=ky=kYe^{i\Omega t} \qquad \ddot{x}+\omega_n^2 x=\frac{k}{m}Ye^{i\Omega t}=\omega_n^2 Ye^{i\Omega t}$$

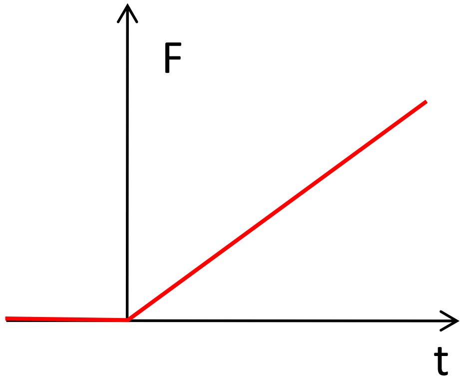
$$x(t)=Xe^{i\Omega t} \qquad \dot{x}(t)=i\Omega Xe^{i\Omega t}$$

$$\ddot{x}(t)=-\Omega^2 Xe^{i\Omega t}$$

$$-\Omega^2 Xe^{i\Omega t}+\omega_n^2 Xe^{i\Omega t}=\omega_n^2 Ye^{i\Omega t}$$

$$X=\frac{\omega_n^2 Y}{\omega_n^2-\Omega^2}=\frac{Y}{1-\frac{\Omega^2}{\omega_n^2}}$$

OSCILLAZIONE SISTEMA 1 G.D.L. NON SMORZATO
Sollecitazione con forza variabile "a rampa"



$$m\ddot{x} + kx = Bt$$

$$x(t) = \boxed{A_1 e^{i\omega_n t} + B_1 e^{-i\omega_n t}} + \boxed{\frac{B}{k} t}$$

Integrale generale omogenea associata

Integrale particolare non omogenea

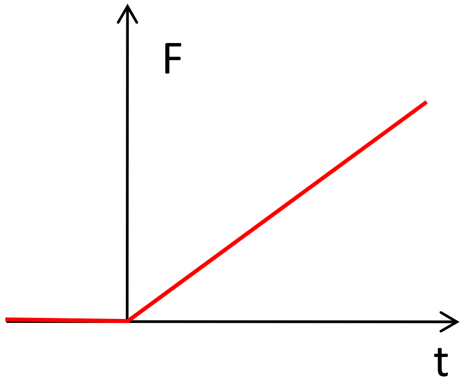
$$x(t) = Xt \qquad \dot{x}(t) = X$$

$$\ddot{x}(t) = 0$$

$$kXt = Bt \quad \Rightarrow \quad X = \frac{B}{k}$$

OSCILLAZIONE SISTEMA 1 G.D.L. NON SMORZATO

Sollecitazione con forza variabile “a rampa”



$$m\ddot{x} + kx = Bt$$

$$x(t) = A_1 e^{i\omega_n t} + B_1 e^{-i\omega_n t} + \frac{B}{k}t$$

Condizioni iniziali $\begin{cases} x(0) = 0 \\ \dot{x}(0) = 0 \end{cases}$

$$\left.\begin{aligned} x(0) &= A_1 + B_1 = 0 \\ \dot{x}(0) &= i\omega_n A_1 - i\omega_n B_1 + \frac{B}{k} \end{aligned}\right\} \Rightarrow \begin{cases} A_1 = i\dfrac{B}{2k\omega_n} \\ B_1 = -i\dfrac{B}{2k\omega_n} \end{cases}$$

$$x(t) = \frac{B}{k}\left(i\frac{e^{i\omega_n t}}{2\omega_n} - i\frac{e^{-i\omega_n t}}{2\omega_n} + t \right) = \frac{B}{k}\left(\frac{i}{2\omega_n} 2iSin(\omega_n t) + t \right) = \frac{B}{k}\left(t - \frac{Sin(\omega_n t)}{\omega_n} \right)$$

OSCILLAZIONE SISTEMA 1 G.D.L. NON SMORZATO
Sollecitazione con forza variabile "a rampa"

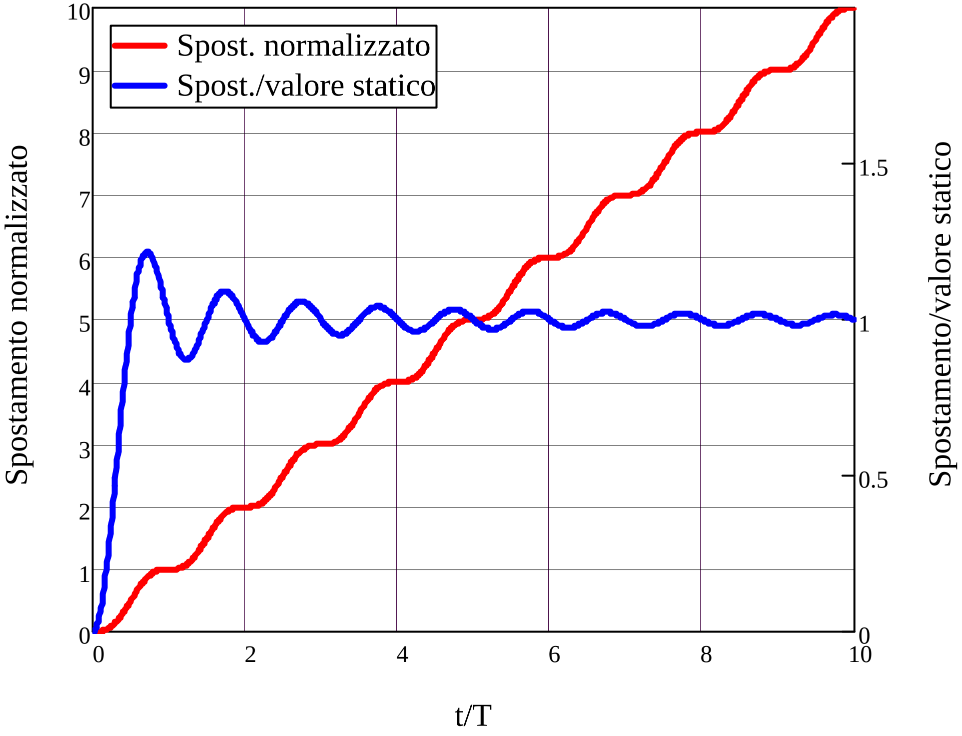
$$x(t)=\frac{B}{k}\left(t-\frac{Sin(\omega_n t)}{\omega_n}\right)=\frac{BT}{k}\left(\frac{t}{T}-\frac{Sin(2\pi\frac{t}{T})}{2\pi}\right)$$

Spostamento normalizzato

$$\frac{x(t)}{\frac{BT}{k}}$$

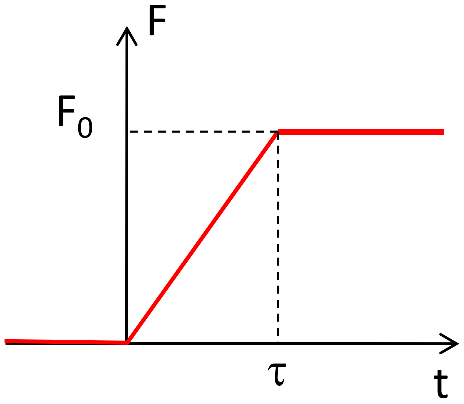
Spostamento/valore statico

$$\frac{x(t)}{\frac{Bt}{k}}=1-\frac{T}{t}\frac{Sin(2\pi\frac{t}{T})}{2\pi}$$



OSCILLAZIONE SISTEMA 1 G.D.L. NON SMORZATO

Sollecitazione con forza variabile a gradino con rampa iniziale



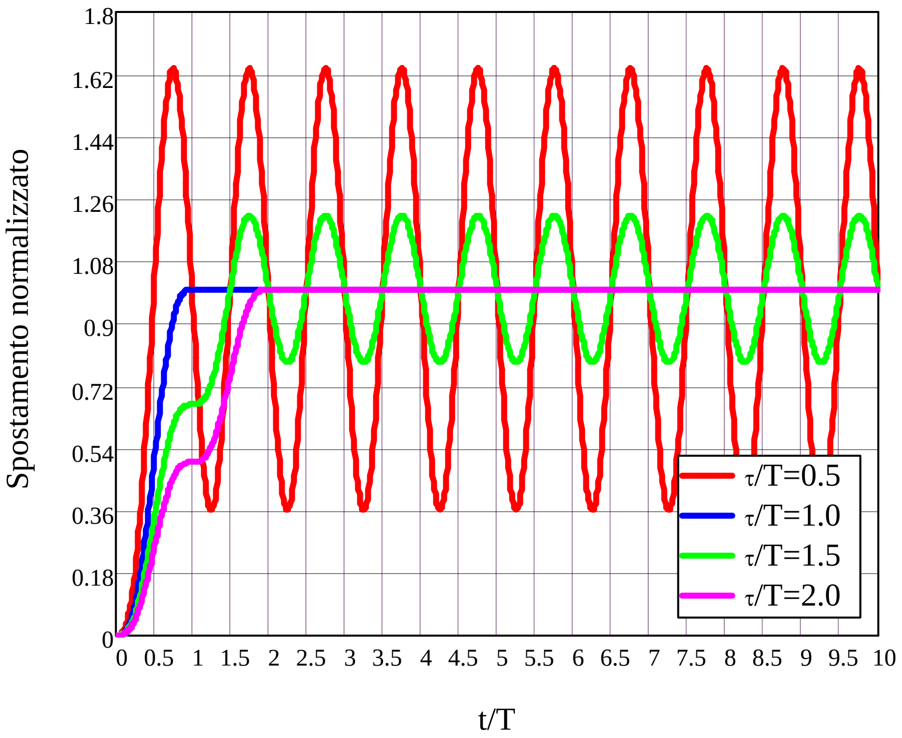
$$\begin{cases} F=\dfrac{F_0}{\tau}t=Bt & 0\le t\le\tau \\ F=Bt-B(t-\tau)=F_0 & \tau<t \end{cases}$$

$$\begin{cases} 0\le t\le\tau & x(t)=\dfrac{B}{k}\left(t-\dfrac{Sin(\omega_n t)}{\omega_n}\right) \\ \tau<t & x(t)=\dfrac{B}{k}\left(t-\dfrac{Sin(\omega_n t)}{\omega_n}\right)-\dfrac{B}{k}\left((t-\tau)-\dfrac{Sin(\omega_n(t-\tau))}{\omega_n}\right)= \\ & =\dfrac{B}{k}\left(\tau-\dfrac{Sin(\omega_n t)}{\omega_n}+\dfrac{Sin(\omega_n(t-\tau))}{\omega_n}\right) \end{cases}$$

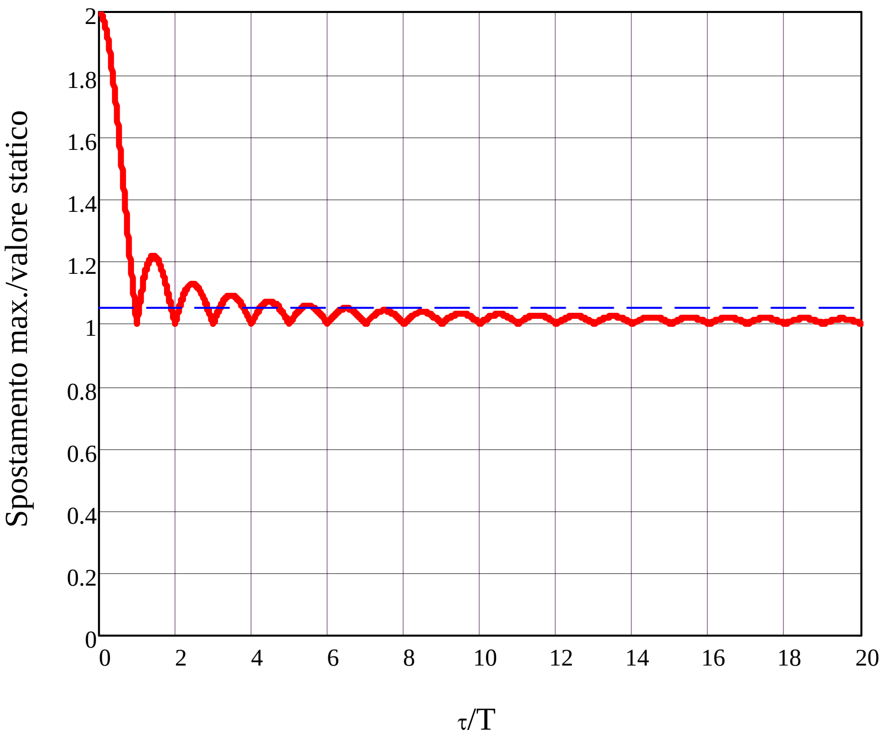
OSCILLAZIONE SISTEMA 1 G.D.L. NON SMORZATO
Sollecitazione con forza variabile a gradino con rampa iniziale

$$\begin{cases} 0 \le t \le \tau \qquad \dfrac{x(t)k}{F_0} = \dfrac{T}{\tau} . \left(\dfrac{t}{T} - . \dfrac{Sin(2\pi \frac{t}{T})}{2\pi} \right) \\ \tau < t \qquad \dfrac{x(t)k}{F_0} = 1 - \dfrac{Sin(\omega_n t)}{\tau\omega_n} + \dfrac{Sin(\omega_n (t-\tau))}{\tau\omega_n} = \\ \qquad\qquad = 1 + \dfrac{T}{\tau} \left(\dfrac{Sin(\omega_n (t-\tau))}{2\pi} - \dfrac{Sin(\omega_n t)}{2\pi} \right) \end{cases}$$

OSCILLAZIONE SISTEMA 1 G.D.L. NON SMORZATO
Sollecitazione con forza variabile a gradino con rampa iniziale

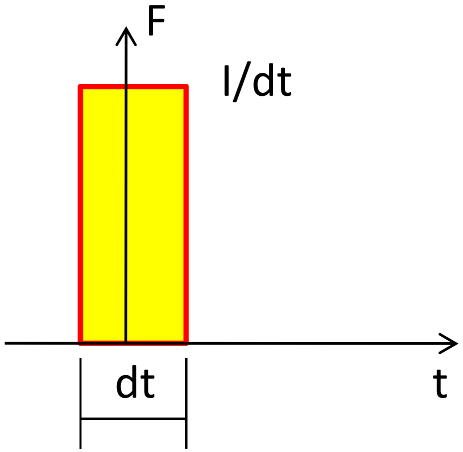


OSCILLAZIONE SISTEMA 1 G.D.L. NON SMORZATO
Sollecitazione con forza variabile a gradino con rampa iniziale



OSCILLAZIONE SISTEMA 1 G.D.L. NON SMORZATO

Sollecitazione con impulso I al tempo t=0



$$F = \lim_{dt \to 0} \frac{I}{dt}$$

$$I = mv \quad \Rightarrow \quad v = \frac{I}{m}$$

Condizioni iniziali

$$\begin{cases} x(0) = 0 \\ \dot{x}(0) = v \end{cases}$$

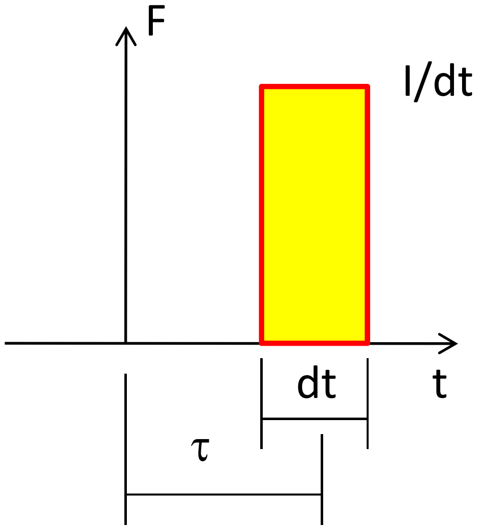
$$x(t) = A_1 e^{i\omega_n t} + B_1 e^{-i\omega_n t}$$

$$\left.\begin{aligned} x(0) &= A_1 + B_1 = 0 \\ \dot{x}(0) &= i\omega_n A_1 - i\omega_n B_1 = \frac{I}{m} \end{aligned}\right\} \Rightarrow \begin{cases} A_1 = -i\dfrac{I}{2m\omega_n} \\ B_1 = i\dfrac{I}{2m\omega_n} \end{cases}$$

$$x(t) = \frac{I}{2\omega_n m}\left(ie^{i\omega_n t} - ie^{-i\omega_n t}\right) = \frac{I}{2\omega_n m}\left(i2iSin(\omega_n t)\right) = \frac{ISin(\omega_n t)}{2\omega_n m}$$

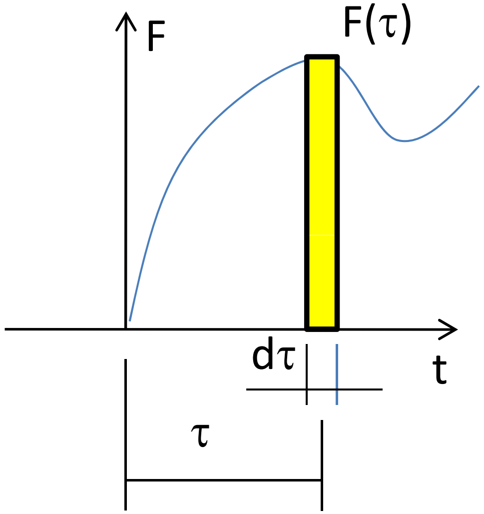
OSCILLAZIONE SISTEMA 1 G.D.L. NON SMORZATO

Sollecitazione con impulso I al tempo t=τ



$$\begin{cases} x(t) = \dfrac{ISin(\omega_n(t-\tau))}{2\omega_n m} & t \geq \tau \\ x(t) = 0 & t < \tau \end{cases}$$

La forza di andamento generico può essere vista come una successione di impulsi di valore F(τ) dτ.

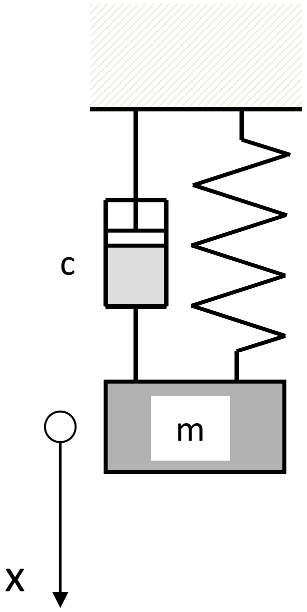


$$x(t) = \frac{1}{2\omega_n m} \int_0^t F(\tau) Sin(\omega_n(t-\tau)) \cdot d\tau$$

Integrale di convoluzione o di Duhamel

OSCILLAZIONE LIBERA SISTEMA 1 G.D.L. SMORZATO

Sistema ad 1 g.d.l.



$$m\ddot{x} + c\dot{x} + kx = 0$$

$$\Downarrow$$

$$x(t) = A_1 \cdot e^{a_1 t} + A_2 \cdot e^{a_2 t}$$

$$\Downarrow$$

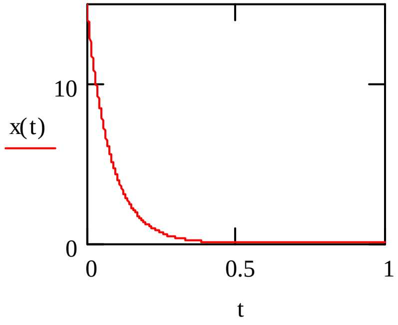
$$a^2 + \frac{c}{m}a + \frac{k}{m} = 0$$

$$\Delta = \frac{c^2}{m^2} - 4\frac{k}{m} = 0 \quad \rightarrow \quad c = c_{cr} = 2\sqrt{km}$$

$$c > c_{cr} \quad \rightarrow \quad \Delta > 0$$

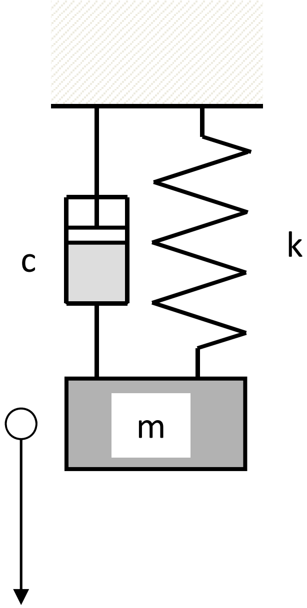
$$a_1,\ a_2 \quad reali < 0$$

$$a_{1,2} = -\frac{c}{2m} \pm \frac{1}{2}\sqrt{\frac{c^2}{m^2} - 4\frac{k}{m}}$$



OSCILLAZIONE LIBERA SISTEMA 1 G.D.L. SMORZATO

Sistema ad 1 g.d.l. $c < c_{cr} \quad \rightarrow \quad \Delta < 0$



$$a_1,\ a_2\ \ complesse\ \ coniugate = -\frac{c}{2m} \pm i \cdot \frac{1}{2}\sqrt{4\frac{k}{m} - \frac{c^2}{m^2}}$$

$$\xi = \frac{c}{c_{cr}}$$

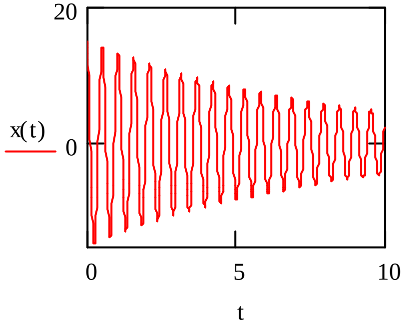
$$\frac{c}{2m} = \frac{c\sqrt{k}}{2m\sqrt{k}} = \frac{c}{2\sqrt{km}}\sqrt{\frac{k}{m}} = \frac{c}{c_{cr}}\omega_n = \xi\omega_n$$

$$\frac{1}{2}\sqrt{4\frac{k}{m} - \frac{c^2}{m^2}} = \sqrt{\frac{k}{m}\left(1 - \frac{c^2}{4mk}\right)} = \sqrt{\frac{k}{m}}\sqrt{1 - \frac{c^2}{c_{cr}^2}} = \omega_n\sqrt{1-\xi^2} = \omega_s$$

$$a_{1,2} = -\xi\omega_n \pm i\omega_s$$

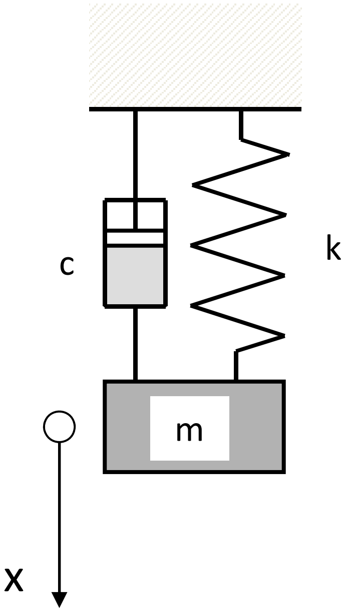
$$x(t) = A_1 e^{(-\xi\omega_n + i\omega_s)t} + B_1 e^{(-\xi\omega_n - i\omega_s)t} = e^{-\xi\omega_n t}\left(A_1 e^{i\omega_s t} + B_1 e^{-i\omega_s t}\right)$$

$$x(t) = e^{-\xi\omega_n t}\left(A\cos(\omega_s t) + B\sin(\omega_s t)\right)$$



OSCILLAZIONE LIBERA SISTEMA 1 G.D.L. SMORZATO

Sistema ad 1 g.d.l.



$$\xi = \frac{c}{c_{cr}}$$

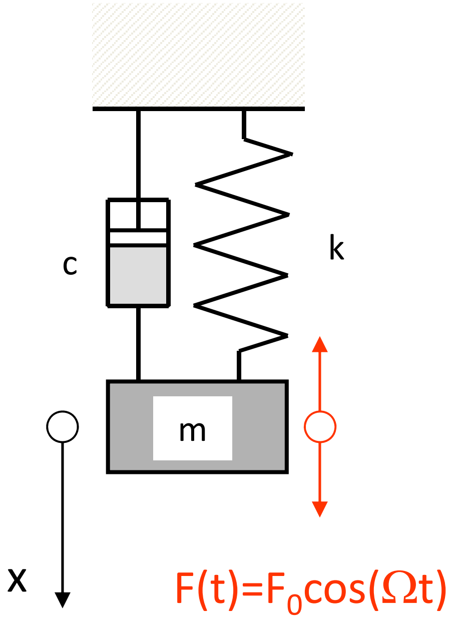
Per la maggior parte dei sistemi meccanici è piuttosto piccolo (< 0.1)

$$\left.\begin{aligned}\omega_s &= \omega_n\sqrt{1-\xi^2}\\ \xi &= 0.1\end{aligned}\right\rangle \omega_s = \omega_n\sqrt{1-0.1^2} = 0.99\omega_n$$

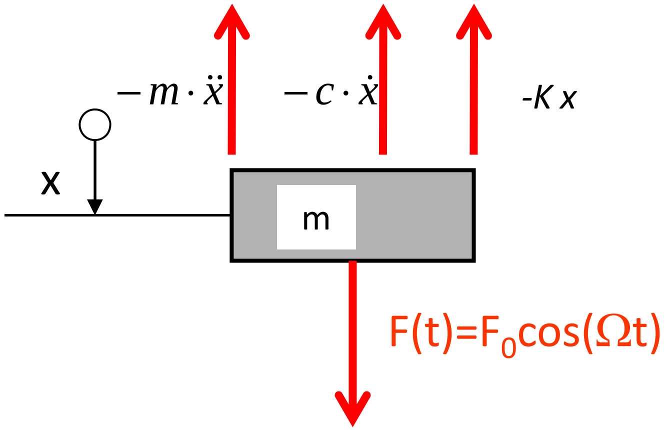
Per questo è solitamente possibile trascurare l’effetto dello smorzamento **sul valore dei modi propri**

OSCILLAZIONE FORZATA SISTEMA 1 G.D.L. SMORZATO

Sistema ad 1 g.d.l.



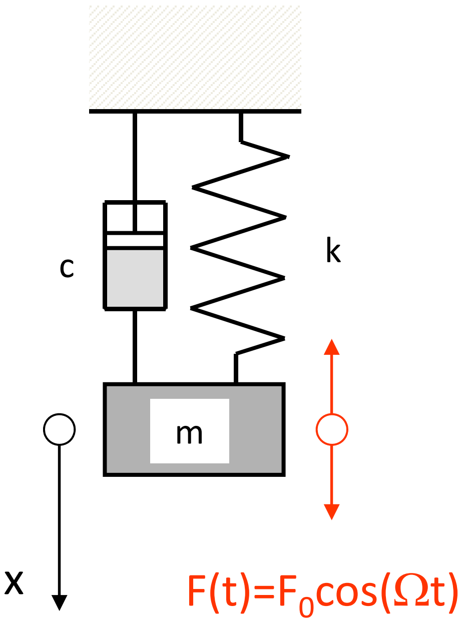
Analisi delle forze agenti



$$m\ddot{x} + c\dot{x} + kx = F_0 \cos(\Omega t)$$

OSCILLAZIONE FORZATA SISTEMA 1 G.D.L. SMORZATO

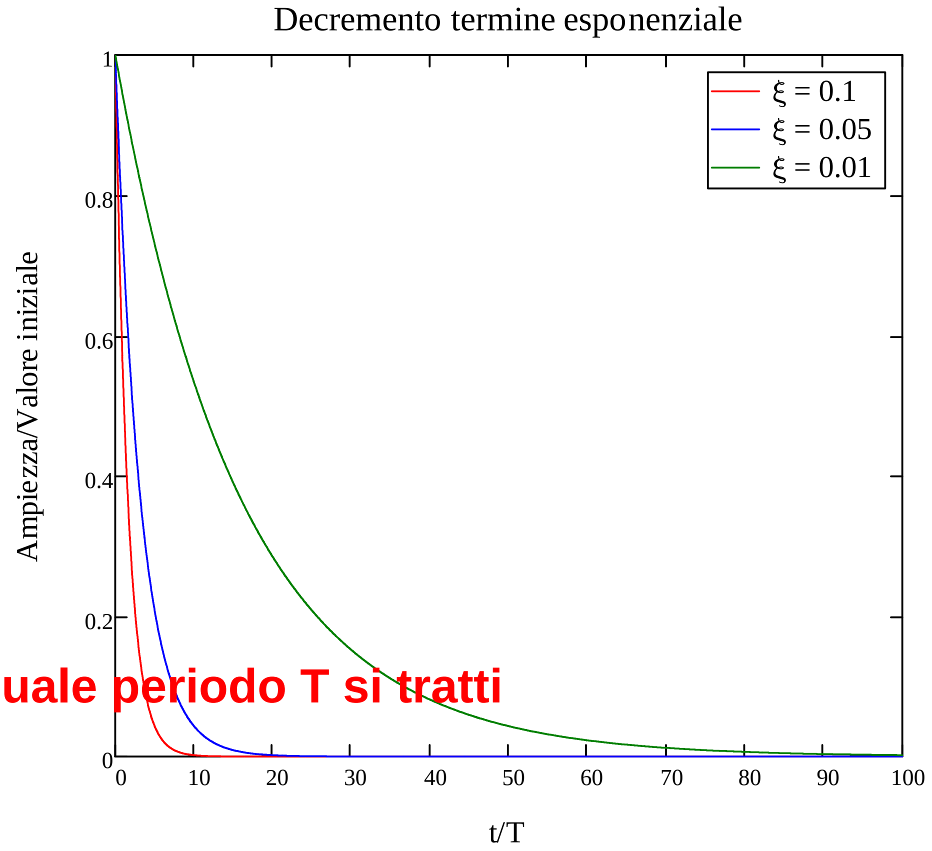
Sistema ad 1 g.d.l.



F(t)=F_0cos(Ωt)

$$m\ddot{x} + c\dot{x} + kx = F_0 \cos(\Omega t)$$

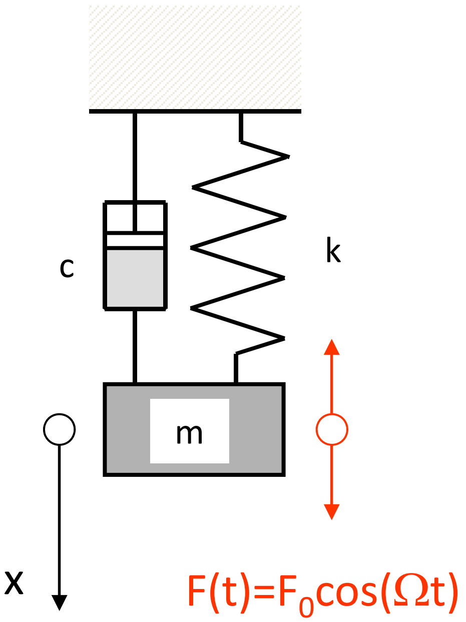
$$x(t) = \boxed{X \cdot \cos(\Omega t - \varphi)} + \boxed{e^{-\xi\omega_n t} A\sin(\omega_s t + \phi)}$$



Nel grafico chiarire di quale periodo T si tratti

OSCILLAZIONE FORZATA SISTEMA 1 G.D.L. SMORZATO

Sistema ad 1 g.d.l.



F(t)=F$_0$cos(Ωt)

$$m\ddot{x} + c\dot{x} + kx = F_0 \cos(\Omega t)$$

$$\Downarrow$$

$$x(t) = \boxed{X \cdot \cos(\Omega t - \varphi)} + \boxed{e^{-\xi\omega_n t} A\sin(\omega_s t + \phi)}$$

$$\Downarrow$$

$$x(t) \cong X \cdot \cos(\Omega t - \varphi) \qquad per\ t > t_{trans}$$

$$X = \frac{F_0}{K} \frac{1}{\sqrt{\left(1 - \frac{\Omega^2}{\omega_n^2}\right)^2 + \left(2\xi \frac{\Omega}{\omega_n}\right)^2}}$$

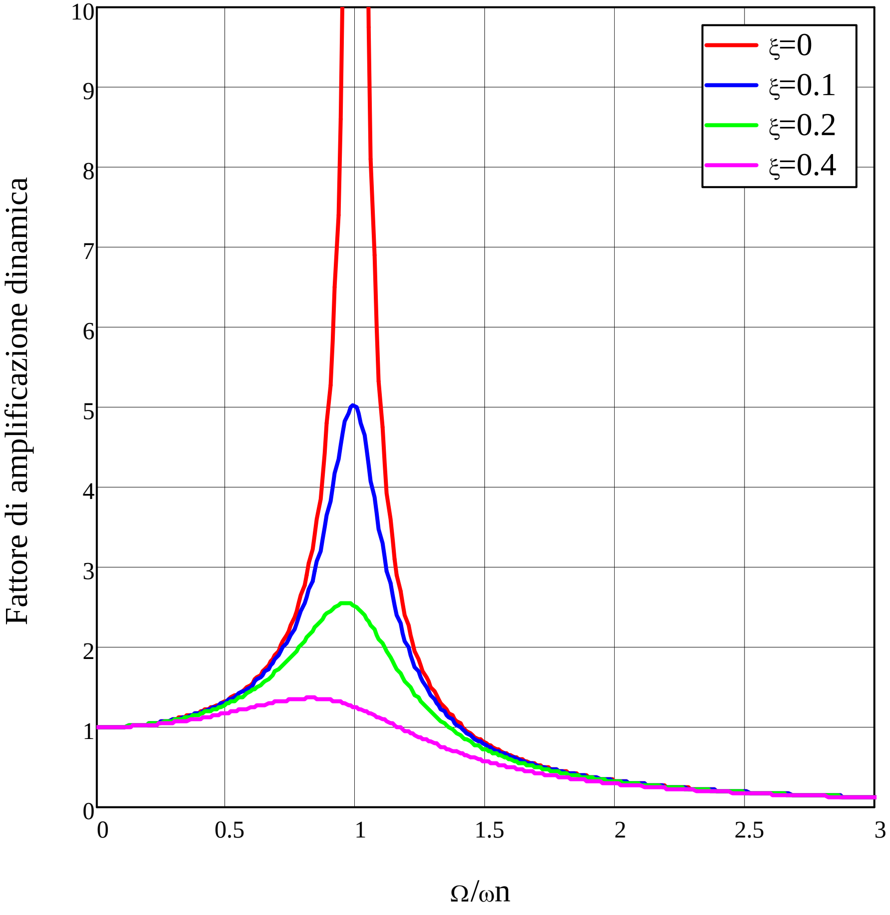
$$\varphi = \arctan\left(\frac{\xi \frac{\Omega}{\omega_n}}{1 - \frac{\Omega^2}{\omega_n^2}}\right)$$

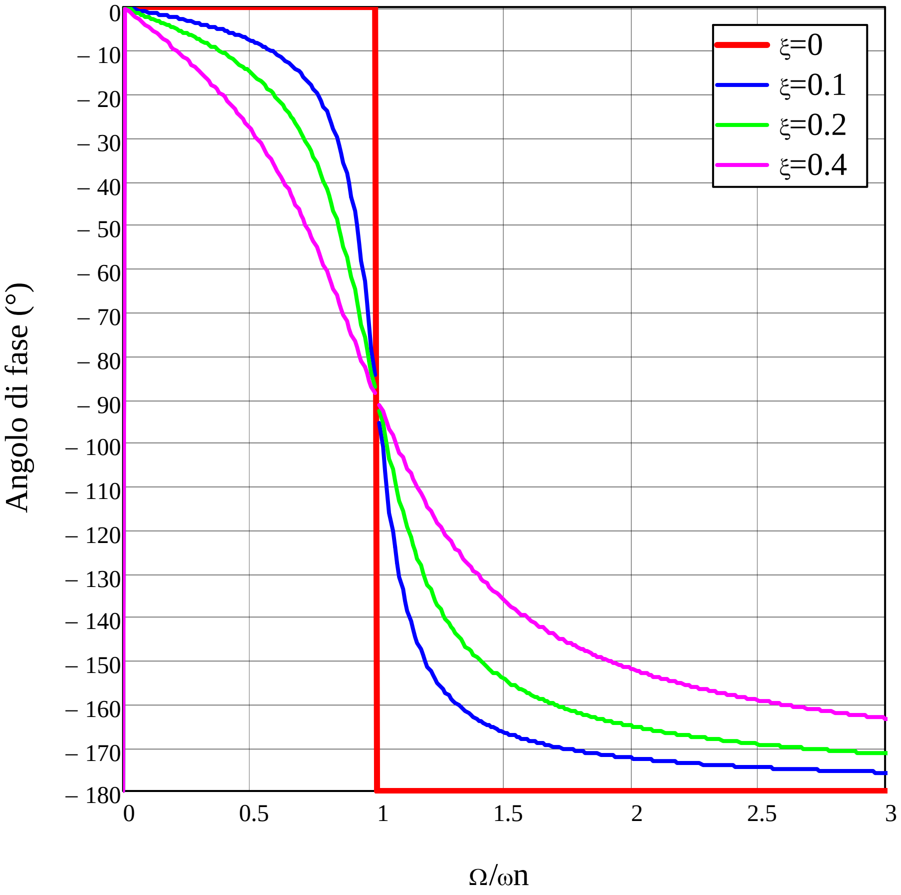
$$\omega_n = \sqrt{\frac{k}{m}}$$

$$\omega_s = \omega_n \sqrt{1 - \xi^2}$$

OSCILLAZIONE FORZATA SISTEMA 1 G.D.L. SMORZATO

$$X = \frac{F_0}{K} \frac{1}{\sqrt{\left(1 - \frac{\Omega^2}{\omega_n^2}\right)^2 + \left(2\xi \frac{\Omega}{\omega_n}\right)^2}}$$





$$\varphi = \arctan\left(\frac{-2\xi \frac{\Omega}{\omega_n}}{1 - \frac{\Omega^2}{\omega_n^2}}\right)$$

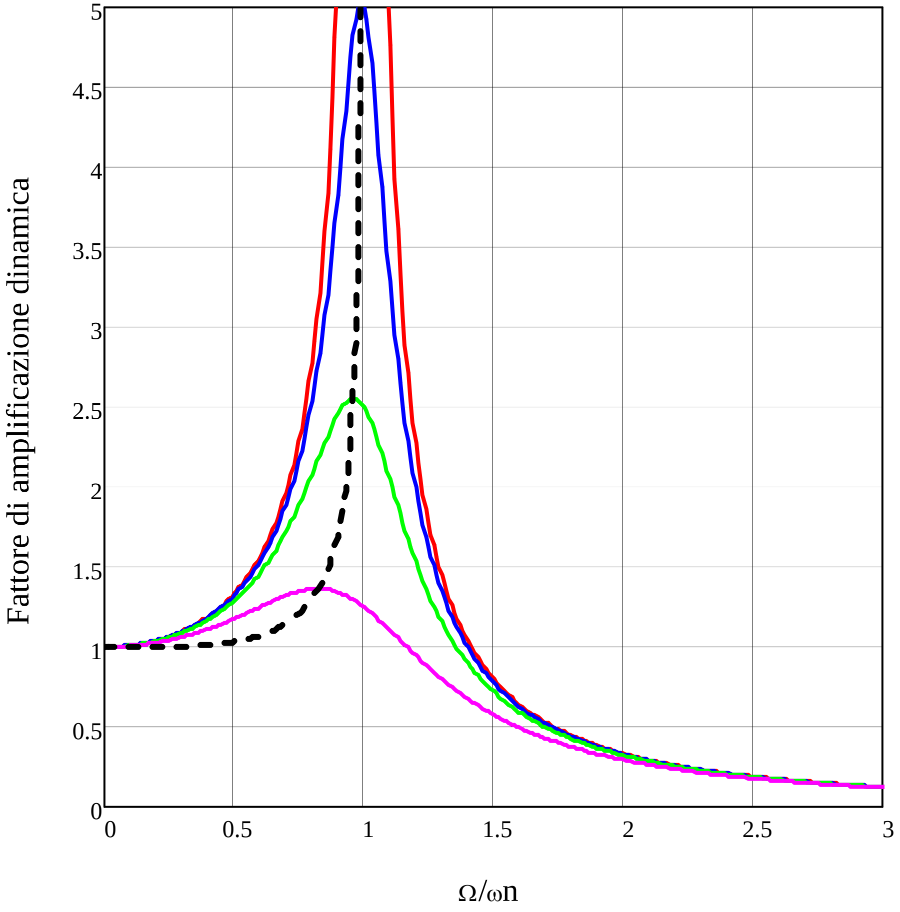
OSCILLAZIONE FORZATA SISTEMA 1 G.D.L. SMORZATO

Rapporto di frequenza per il quale si ha il massimo valore del fattore di amplificazione dinamica:

$$\left(\frac{\Omega}{\omega_n}\right)_{\max} = \sqrt{1-2\xi^2}$$

Massimo valore del fattore di amplificazione dinamica:

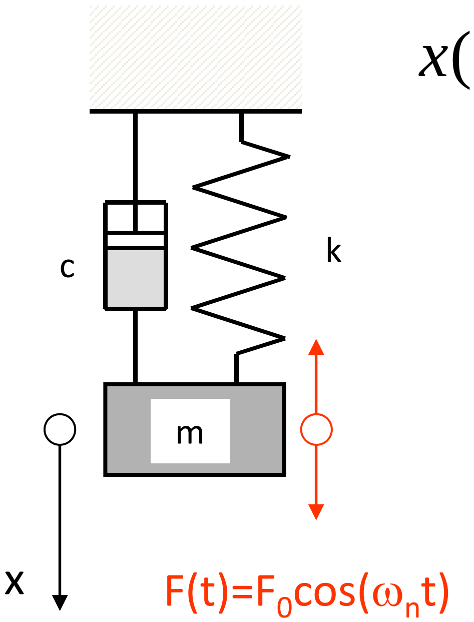
$$D_{\max} = \sqrt{1-\left(\frac{\Omega}{\omega_n}\right)^4}$$



OSCILLAZIONE FORZATA IN RISONANZA SISTEMA 1 G.D.L. SMORZATO

Sistema ad 1 g.d.l.

$$\ddot{x} + \xi\omega_n\dot{x} + \omega_n^2 x = \frac{F_0}{m} e^{i\omega_n t}$$



$$x(t) = \boxed{Xe^{i\omega_n t}}$$

Integrale particolare
non omogenea

$$\dot{x}(t) = i\omega_n X e^{i\omega_n t}$$

$$\ddot{x}(t) = -\omega_n^2 X e^{i\omega_n t}$$

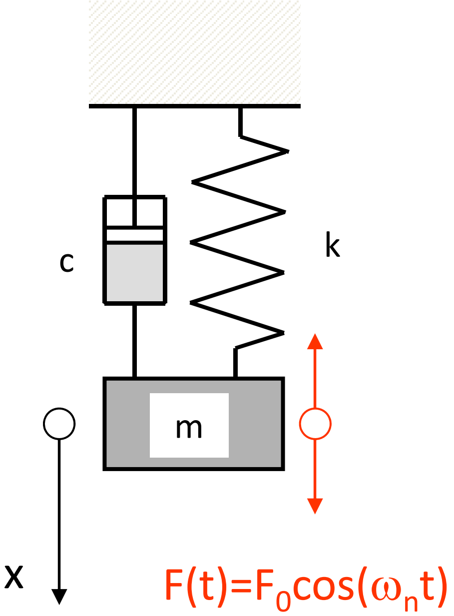
$$-\omega_n^2 X e^{i\omega_n t} + i2\xi\omega_n^2 X e^{i\omega_n t} + \omega_n^2 X e^{i\omega_n t} = \frac{F_0}{m} e^{i\omega_n t}$$

$$-\omega_n^2 X + i2\xi\omega_n^2 X + \omega_n^2 X = \frac{F_0}{m}$$

$$X = \frac{F_0}{m}\frac{1}{2i\xi\omega_n^2} = \frac{F_0}{km}\frac{k}{2i\xi\omega_n^2} = -i\frac{F_0}{k}\frac{1}{2\xi}$$

OSCILLAZIONE FORZATA IN RISONANZA SISTEMA 1 G.D.L. SMORZATO

Sistema ad 1 g.d.l.



$$x(t)=e^{-\xi\omega_n t}(A_1 e^{i\omega_s t}+B_1 e^{-i\omega_s t})-i\frac{F_0}{k}\frac{1}{2\xi}e^{-i\omega_n t}$$

Condizioni iniziali

$$\begin{cases} x(0)=0 \\ \dot{x}(0)=0 \end{cases}$$

$$x(0)=A_1+B_1-i\frac{F_0}{k2\xi}=0$$

$$\dot{x}(0)=-\xi\omega_n(A_1+B_1)+i\omega_n\sqrt{1-\xi^2}(A_1-B_1)-\frac{\omega_n F_0}{2k\xi}=0$$

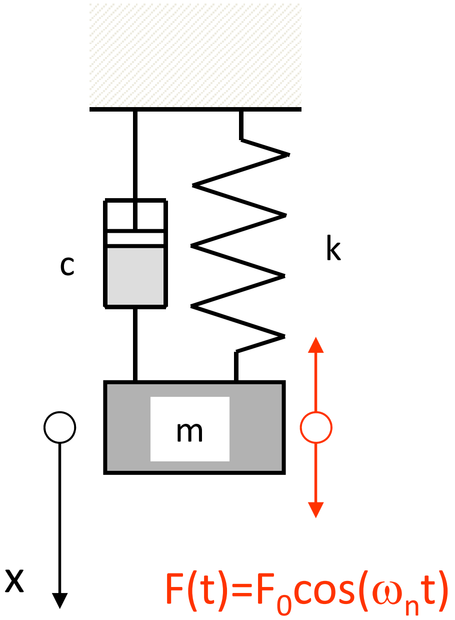
$$B_1=\frac{F_0}{4k\xi\sqrt{1-\xi^2}}\left[-\xi+i\left(1+\sqrt{1-\xi^2}\right)\right]$$

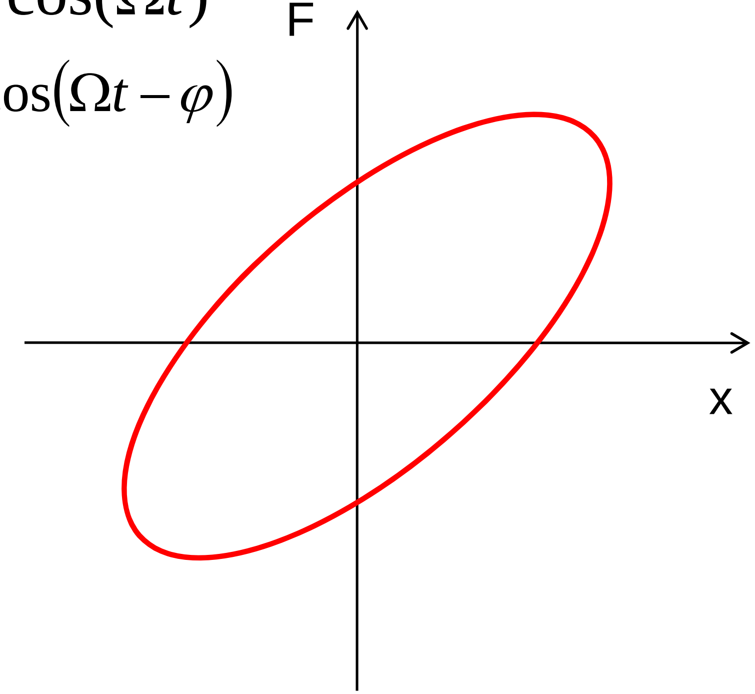
$$A_1=\frac{F_0}{4k\xi\sqrt{1-\xi^2}}\left[\xi-i\left(1-\sqrt{1-\xi^2}\right)\right]$$

LAVORO DI UNA FORZA ARMONICA IN UN CICLO

$$F(t) = F_0 \cos(\Omega t)$$

$$x(t) = X \cdot \cos(\Omega t - \varphi)$$





LAVORO DI UNA FORZA ARMONICA IN UN CICLO

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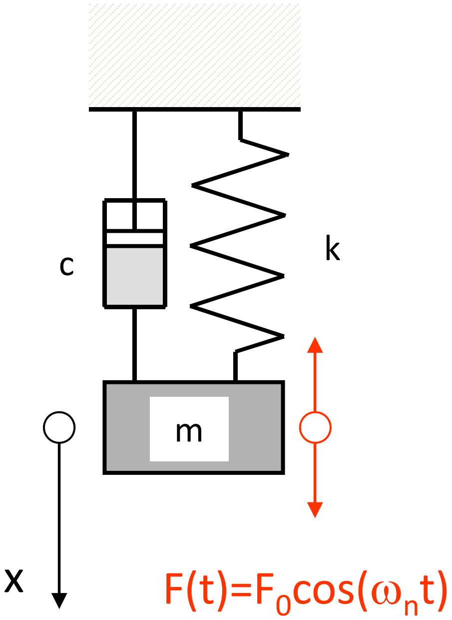
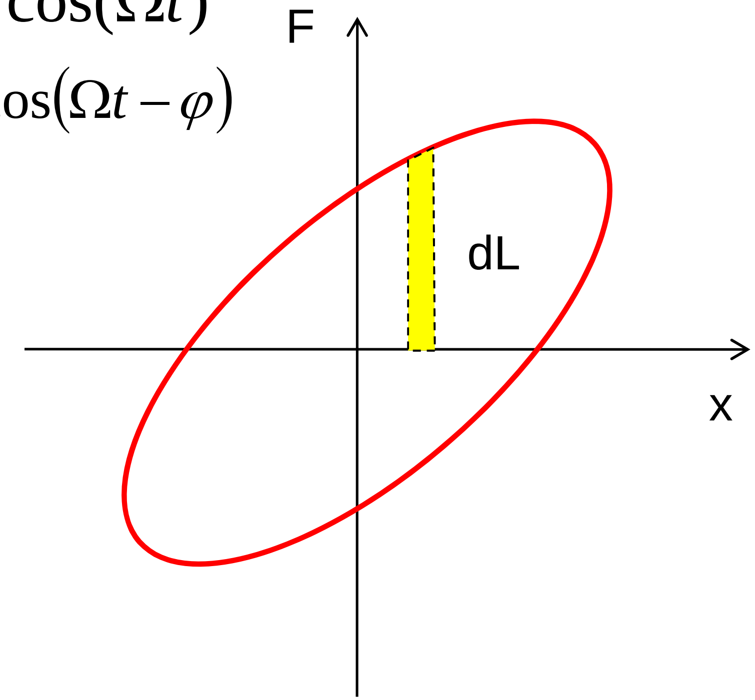


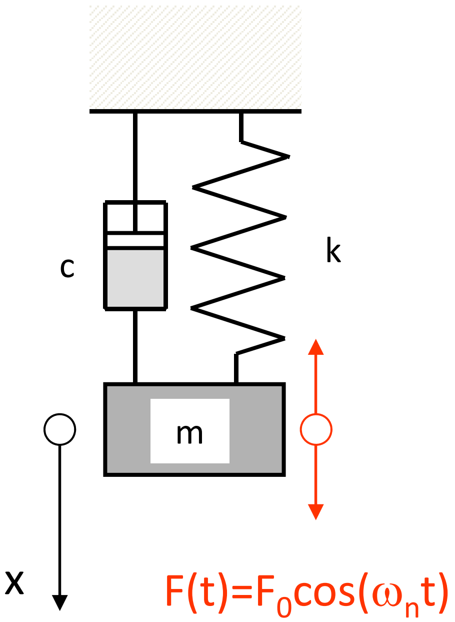


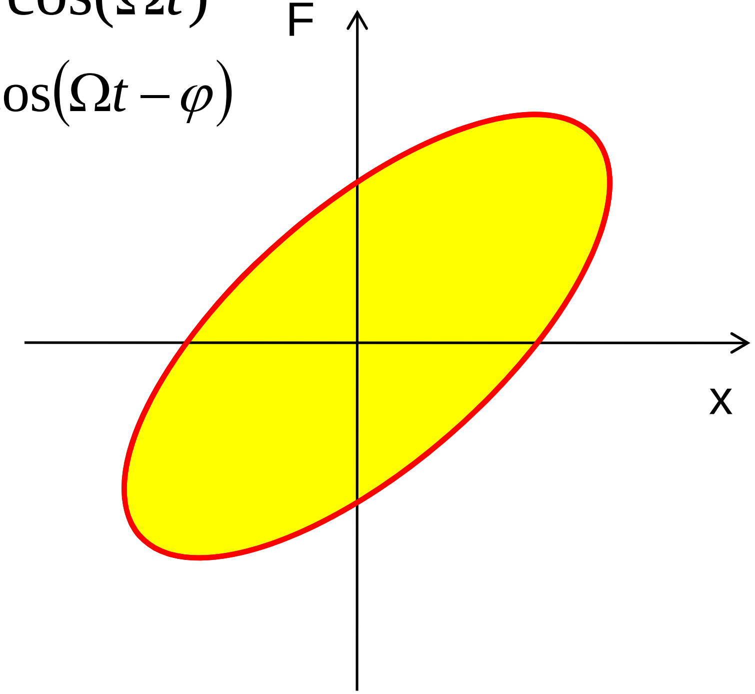
$$dL = F(t)dx = F(t)\dot{x} \cdot dt$$

LAVORO DI UNA FORZA ARMONICA IN UN CICLO

$$F(t) = F_0 \cos(\Omega t)$$

$$x(t) = X \cdot \cos(\Omega t - \varphi)$$

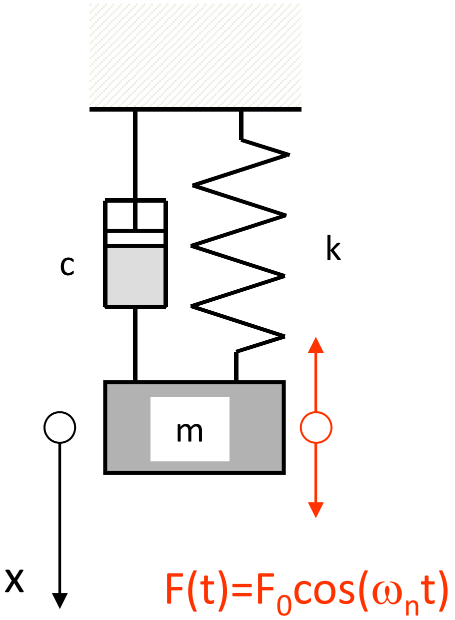




$$dL = F(t)dx = F(t)\dot{x} \cdot dt$$

$$L = \int_0^T F(t)\dot{x} \cdot dt$$

LAVORO DI UNA FORZA ARMONICA IN UN CICLO



$$L = \int_0^T F(t)\dot{x}\cdot dt = \int_0^T F_0\cos(\Omega t)\Omega X\sin(\Omega t - \varphi)\cdot dt$$

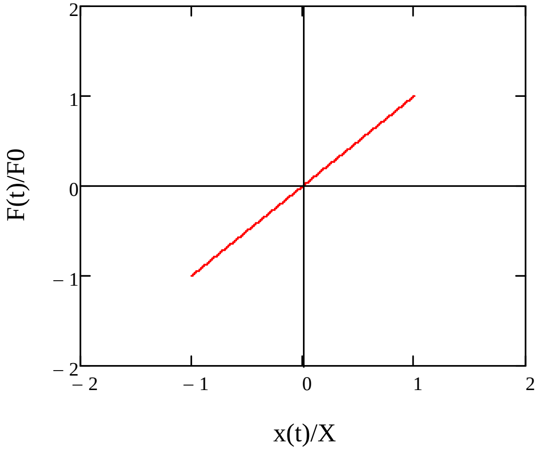
$$= F_0\Omega X\int_0^T \cos(\Omega t)\sin(\Omega t - \varphi)\cdot dt$$

$$= F_0\Omega X\int_0^T \frac{1}{2}\left[\sin(2\Omega t - \varphi) + \sin(-\varphi)\right]\cdot dt$$

$$= \frac{F_0\Omega X}{2}\sin(\varphi)\int_0^T dt \;= \frac{F_0\Omega X}{2}\sin(\varphi)\frac{2\pi}{\Omega} =$$

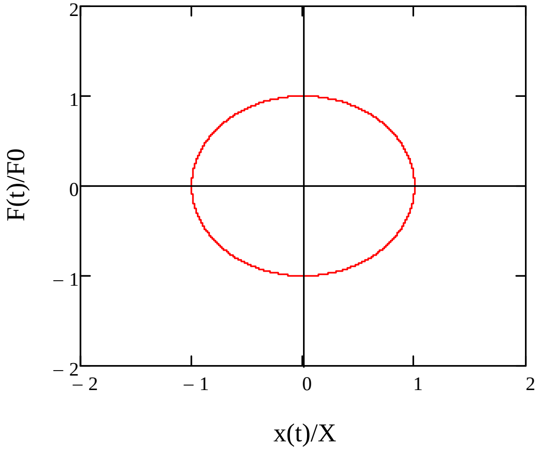
$$= \pi F_0 X\sin(\varphi)$$

LAVORO DI UNA FORZA ARMONICA IN UN CICLO



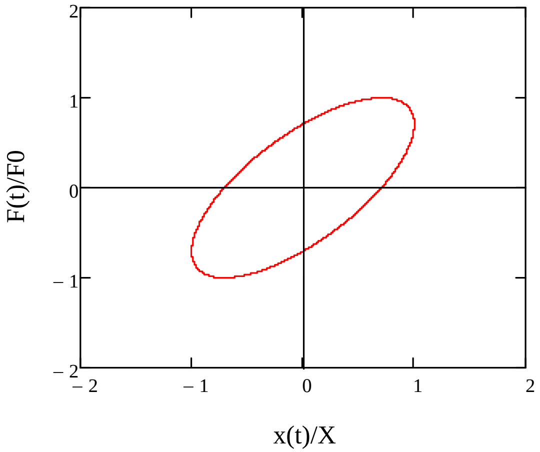
$$\varphi = 0° \; o \; 180°$$

$$L = 0$$



$$\varphi = 90°$$

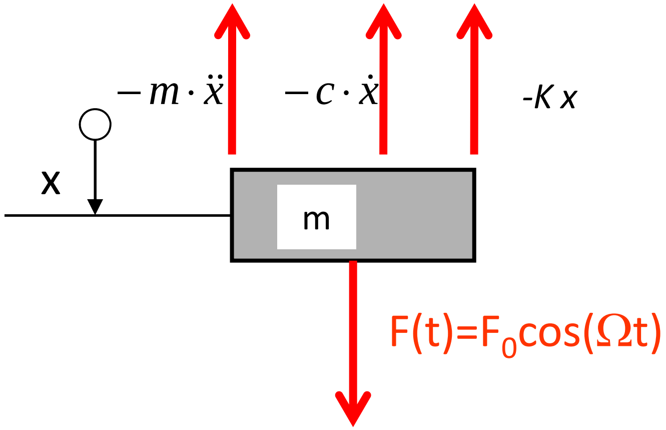
$$L = \pi F_0 X$$



$$\varphi = 45°$$

$$L = \pi F_0 X \, \frac{\sqrt{2}}{2}$$

LAVORO DI UNA FORZA ARMONICA IN UN CICLO



$$x(t) = X\cdot\cos(\Omega t-\varphi)$$

$$\dot{x}(t) = -\Omega X\cdot\sin(\Omega t-\varphi)$$

$$\ddot{x}(t) = -\Omega^2 X\cdot\cos(\Omega t-\varphi)$$

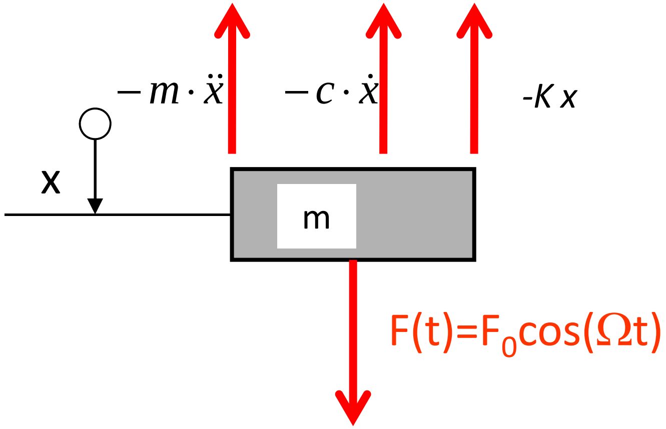
$$Forza\,elastica\;\; molla = -kx \rightarrow\; fase\; con\; x(t) = 180° \rightarrow L_k = 0$$

$$Forza\,smorzatore = -c\dot{x} \rightarrow\; fase\; con\; x(t) = 270° \rightarrow L_c = \pi c\Omega X^2$$

$$Forza\,inerzia = -m\ddot{x} \rightarrow\; fase\; con\; x(t) = 0° \rightarrow L_i = 0$$

$$Lavoro\;\; F\;\; esterna = \pi F_0 X\sin(\varphi) = L_c$$

LAVORO DI UNA FORZA ARMONICA IN UN CICLO



$$\pi F_0 X \sin(\varphi) = \pi c \Omega X^2$$

$$\Downarrow$$

$$X = \frac{F_0}{c\Omega}\sin(\varphi)$$

$$Se\ \Omega = \omega_n \rightarrow \varphi = 90°$$

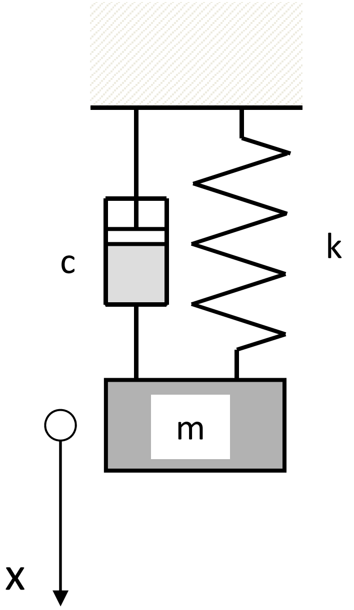
$$X = \frac{F_0}{c\omega_n} = \frac{F_0}{c\sqrt{\frac{k}{m}}} = \frac{F_0}{\frac{c\cdot 2\cdot k}{2\sqrt{km}}} = \frac{F_0}{k}\frac{1}{2\xi}$$

Da soluzione generale

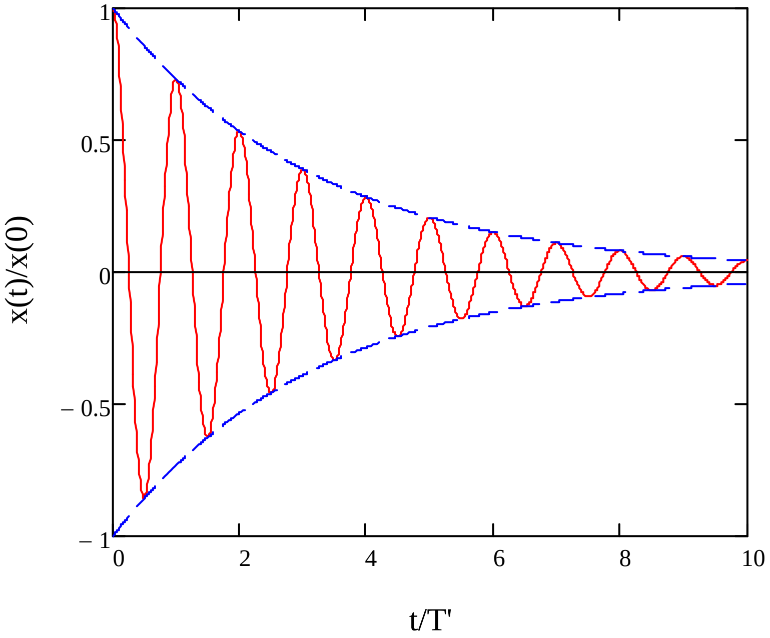
$$X = \frac{F_0}{k}\frac{1}{\sqrt{\left(1-\frac{\Omega^2}{\omega_n^2}\right)^2 + \left(2\xi\frac{\Omega}{\omega_n}\right)^2}} =$$

$$= \frac{F_0}{k}\frac{1}{2\xi}$$

DETERMINAZIONE SPERIMENTALE DELLO SMORZAMENTO RELATIVO
METODO DEL DECREMENTO LOGARITMICO



Si basa sull'andamento delle ampiezze di oscillazione rilevate sulla struttura, in seguito ad una perturbazione iniziale.



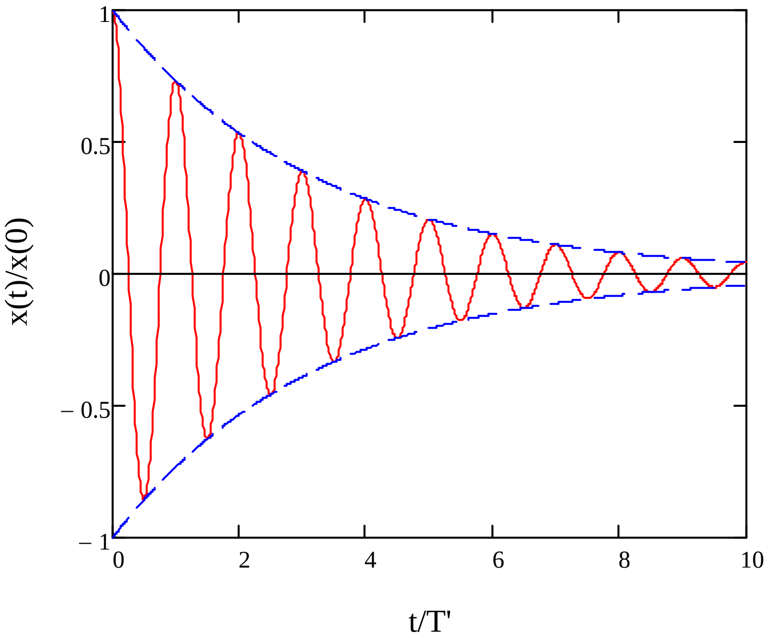
$$x(t) = e^{-\xi\omega_n t}\left(A\cos(\omega_s t) + B\sin(\omega_s t)\right)$$

DETERMINAZIONE SPERIMENTALE DELLO SMORZAMENTO RELATIVO
METODO DEL DECREMENTO LOGARITMICO

Rapporto di ampiezza tra due picchi successivi

$$T' = \frac{2\pi}{\omega_s}$$

$$R = \frac{e^{-\xi\omega_n t}\left(A\cos(\omega_s t) + B\sin(\omega_s t)\right)}{e^{-\xi\omega_n (t+T')}\left(A\cos(\omega_s (t+T')) + B\sin(\omega_s (t+T'))\right)} = \frac{e^{-\xi\omega_n t}}{e^{-\xi\omega_n (t+T')}}$$



Decremento Logaritmico

$$\delta = \ln\left(\frac{e^{-\xi\omega_n t}}{e^{-\xi\omega_n (t+T')}}\right) = \xi\omega_n T' = \xi\omega_n \frac{2\pi}{\omega_s}$$

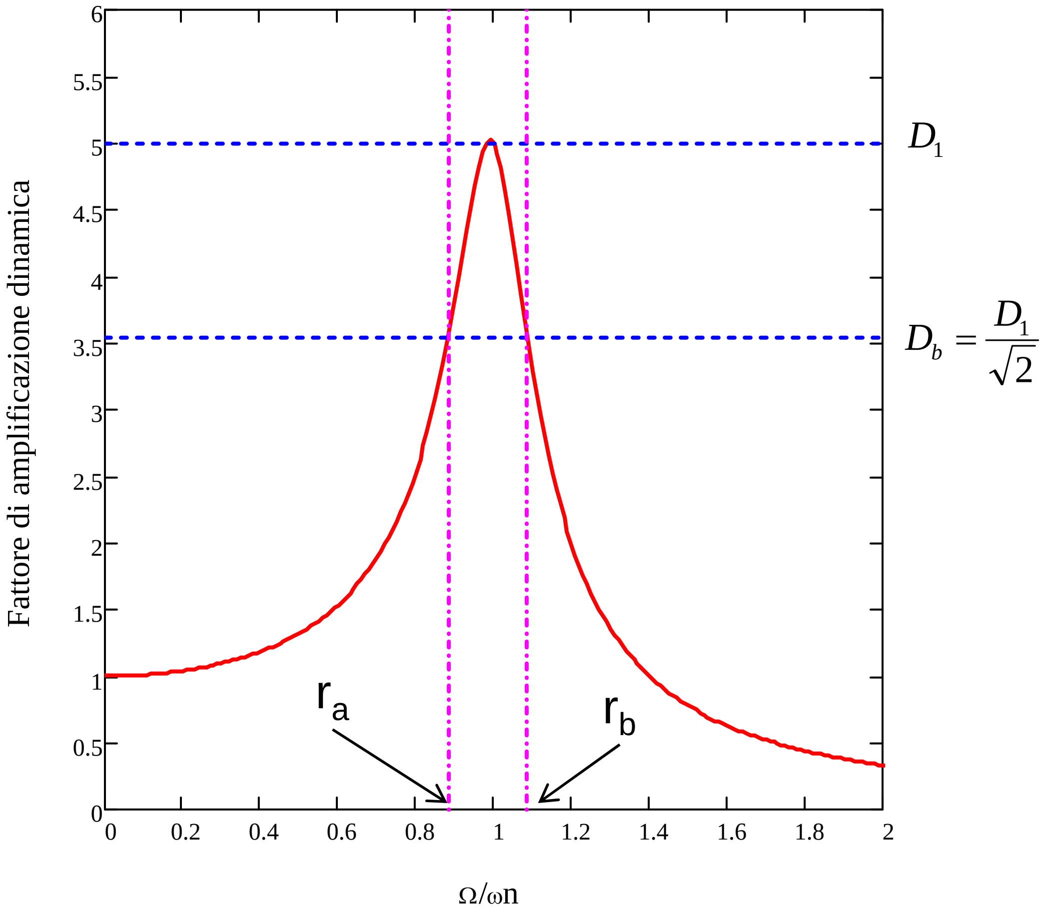
$$= \xi\omega_n \frac{2\pi}{\omega_n\sqrt{1-\xi^2}} = \frac{2\pi\xi}{\sqrt{1-\xi^2}}$$

$$\Downarrow$$

$$\xi = \frac{\delta}{\sqrt{4\pi + \delta^2}}$$

DETERMINAZIONE SPERIMENTALE DELLO SMORZAMENTO RELATIVO
METODO DELLA LARGHEZZA DI BANDA

Si basa sull’andamento del coefficiente di amplificazione dinamica del sistema al variare della frequenza della forzante.



DETERMINAZIONE SPERIMENTALE DELLO SMORZAMENTO RELATIVO METODO DELLA LARGHEZZA DI BANDA

Calcolo di r_a ed r_b

$$D_1 = \frac{1}{2\xi}$$

$$D_b = \frac{D_1}{\sqrt{2}} = \frac{1}{2\sqrt{2}\xi}$$

$$r = \frac{\Omega}{\omega_n}$$

$$D = \frac{1}{\sqrt{\left(1-r^2\right)^2 + \left(2\xi r\right)^2}}$$

$$\frac{1}{\sqrt{\left(1-r^2\right)^2 + 4\xi^2 r^2}} = \frac{1}{2\sqrt{2}\xi}$$

Elevando al quadrato

$$\frac{1}{\left(1-r^2\right)^2 + 4\xi^2 r^2} = \frac{1}{8\xi^2}$$

$$r^4 + 2r^2\left(2\xi^2 - 1\right) + \left(1 - 8\xi^2\right) = 0$$

$$r_{a,b}^2 = 1 - 2\xi^2 \pm \sqrt{\left(2\xi^2 - 1\right)^2 - \left(1 - 8\xi^2\right)} = 1 - 2\xi^2 \pm \sqrt{\left(4\xi^4 - 4\xi^2 + 1\right) - \left(1 - 8\xi^2\right)} = 1 - 2\xi^2 \pm 2\xi\sqrt{1 + \xi^2}$$

DETERMINAZIONE SPERIMENTALE DELLO SMORZAMENTO RELATIVO
METODO DELLA LARGHEZZA DI BANDA

$$r_{a,b}^2 = 1 - 2\xi^2 \pm 2\xi\sqrt{1+\xi^2}$$

Per ξ<<1

$$r_{a,b}^2 \approx 1 - 2\xi^2 \pm 2\xi$$

$$r_{a,b} = \sqrt{1 - 2\xi^2 \pm 2\xi}$$

Per x<<1 si può porre

$$\sqrt{1+x} \approx 1 + \frac{x}{2} + \ldots \quad \Rightarrow \quad r_{a,b} \approx 1 - \xi^2 \pm \xi$$

$$\Downarrow$$

$$r_a = 1 - \xi^2 - \xi \qquad r_b = 1 - \xi^2 + \xi$$

$$\boxed{\xi \approx \frac{r_b - r_a}{2}}$$