



**Corso di**  
**ASPETTI TEORICI ED APPLICATIVI DEL METODO**  
**DEGLI ELEMENTI FINITI – PARTE I**  
**Rev: 03 del 15/03/2012**

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## CONTENUTI DEL CORSO

### LEZIONI

- Basi teoriche del MEF
- Applicazione del MEF a problemi strutturali in campo elastico lineare
- Analisi critica dei risultati di un modello ad EF
- Criteri di modellazione di strutture con il MEF

### ESERCITAZIONI

- Uso del programma ANSYS
- Esempi significativi di applicazione del MEF a problemi strutturali



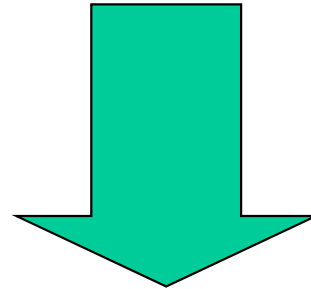
Elasticità

Elettromagnetismo

Fluidodinamica

Termodinamica

Etc...



Sistemi di equazioni differenziali alle derivate parziali

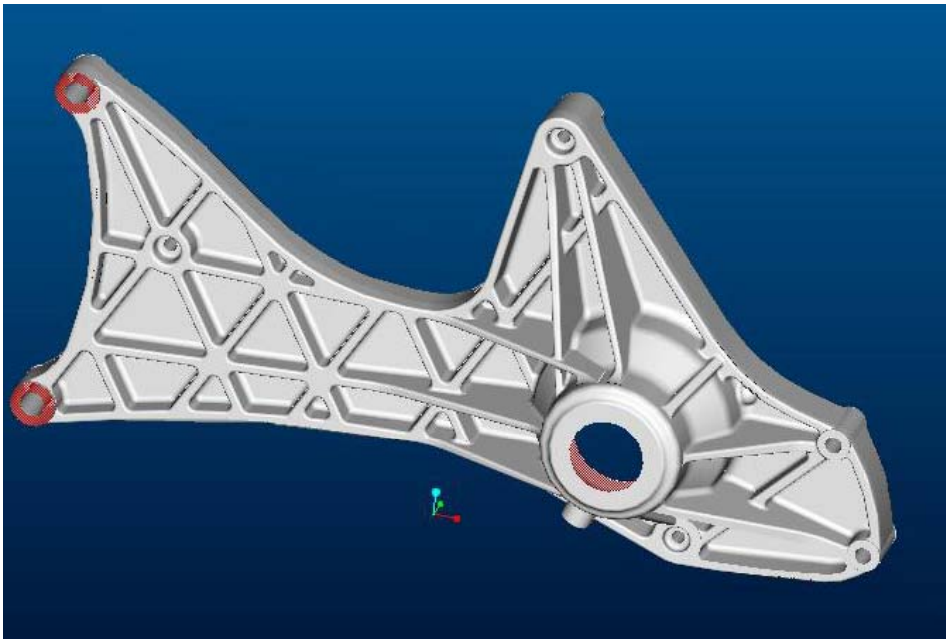
E.qni di Navier

$$\left\{ \begin{array}{l} \nabla^2 u + \frac{1}{1-2\nu} \cdot \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) + \frac{X}{G} = 0 \\ \nabla^2 v + \frac{1}{1-2\nu} \cdot \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) + \frac{Y}{G} = 0 \\ \nabla^2 w + \frac{1}{1-2\nu} \cdot \frac{\partial}{\partial z} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) + \frac{Z}{G} = 0 \end{array} \right.$$

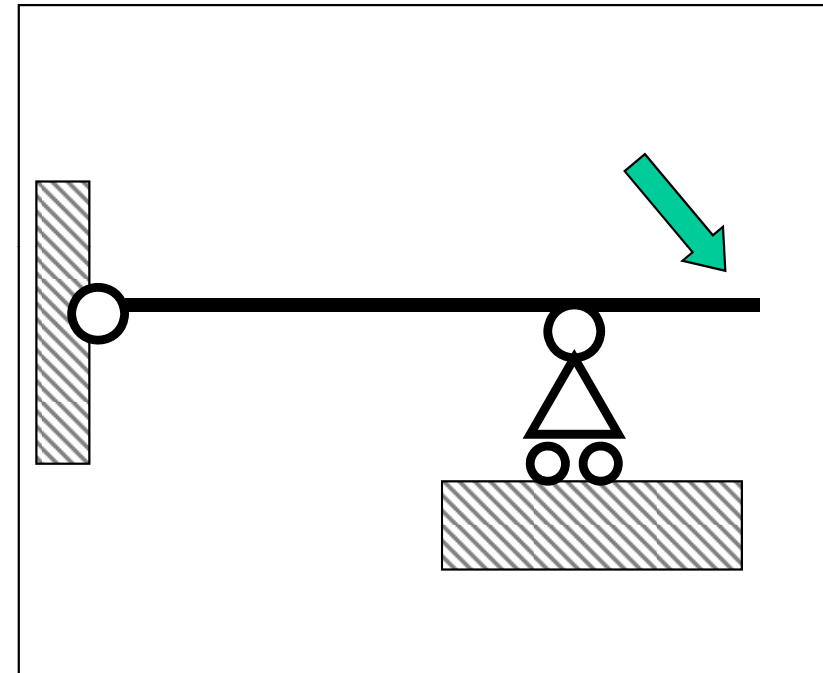
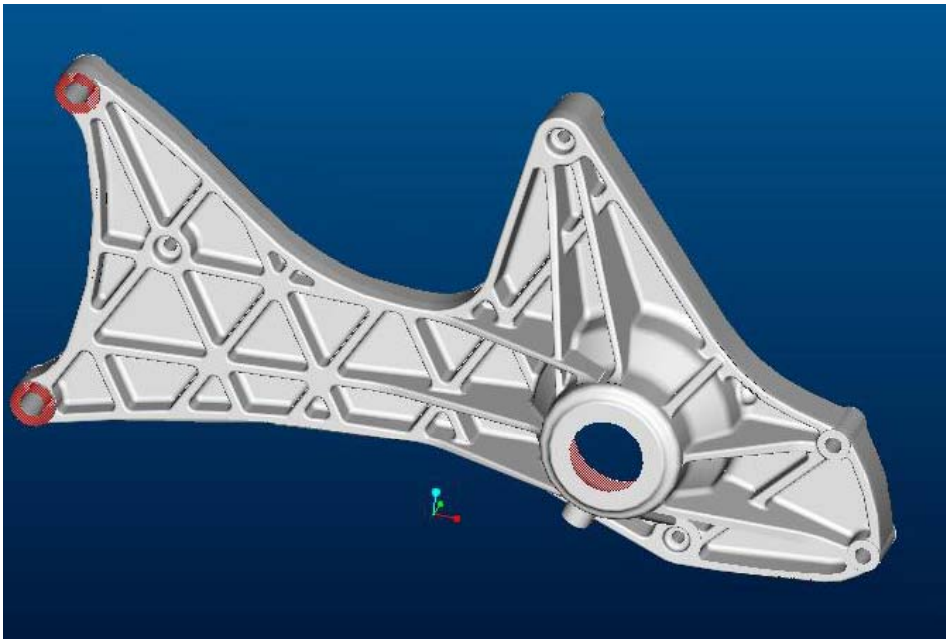


Soluzioni analitiche: solo in casi particolari, introducendo rilevanti semplificazioni (travi, piastre, gusci...)

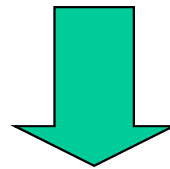
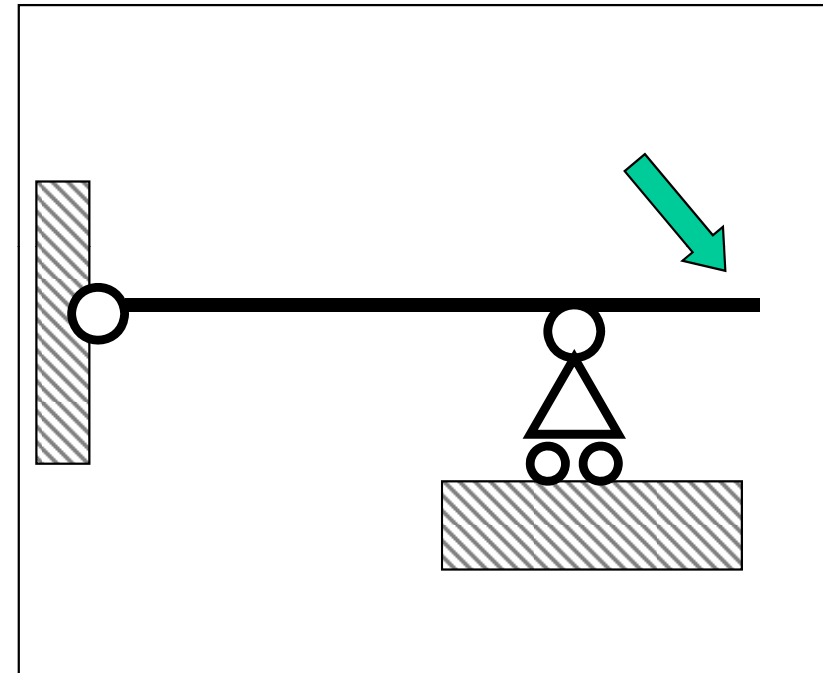
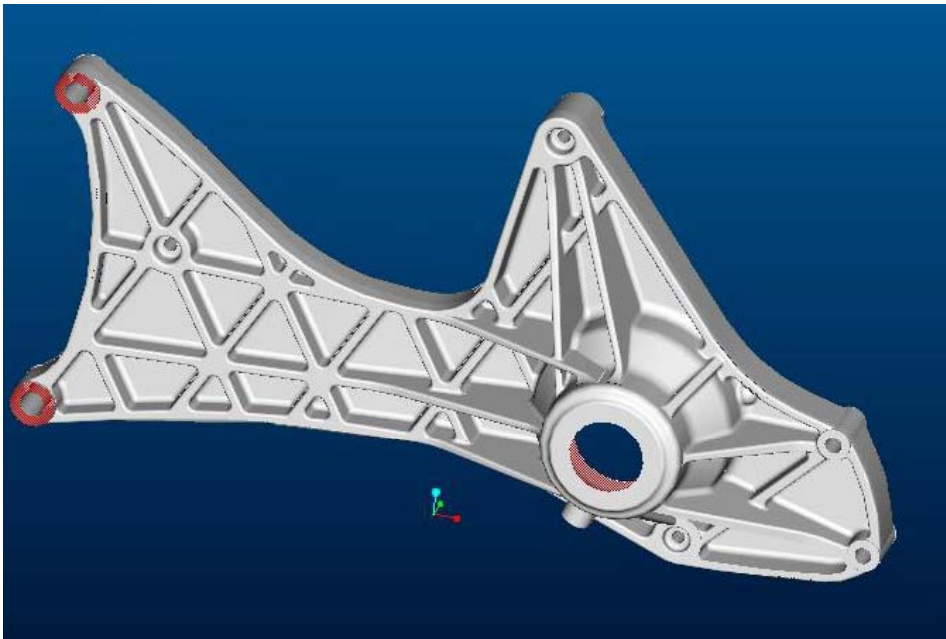
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Sviluppo di tecniche di soluzione **approssimate**



Metodi di soluzione approssimata:

- Differenze finite
- Elementi Finiti
- Elementi di contorno
- Metodi "mesh free"
- ...

Il Metodo degli Elementi Finiti (MEF) è oggi di gran lunga il più diffuso, soprattutto a causa della sua estrema versatilità





Idea centrale del MEF (e delle altre tecniche approssimate):

**Problema originale:** determinare le f.ni incognite  $u$ ,  $v$ ,  $w$



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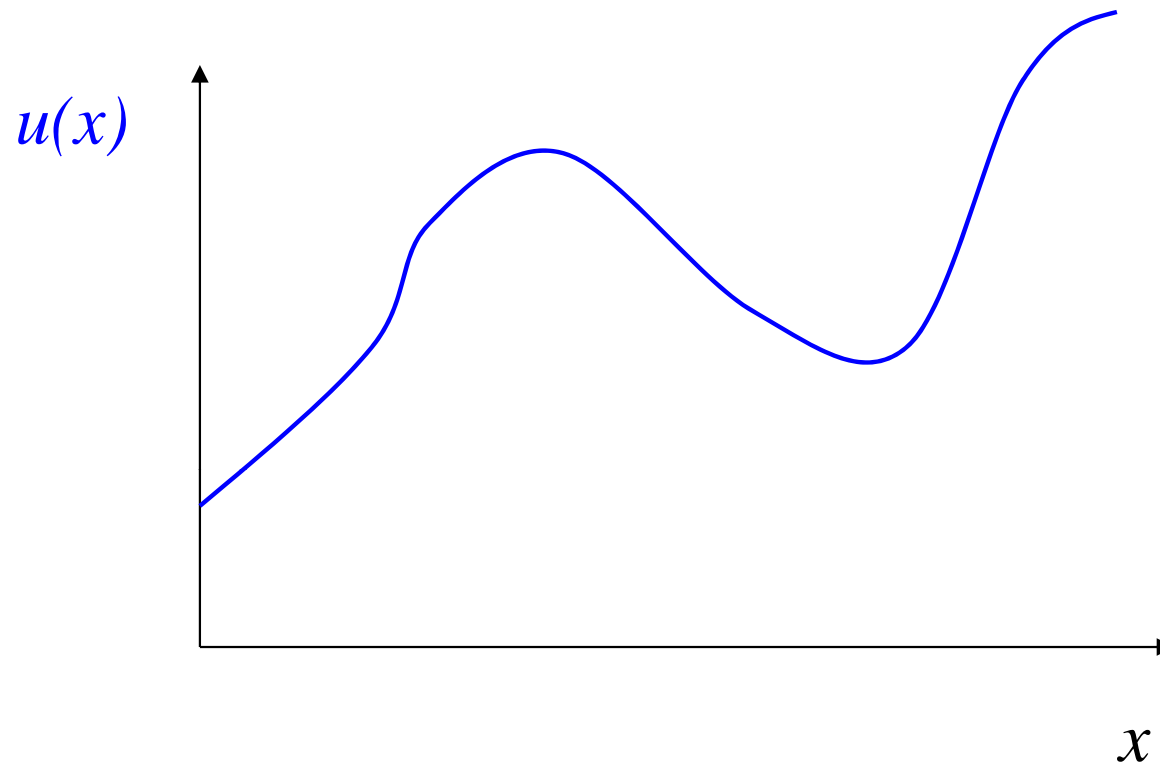
**Problema originale:** determinare le f.ni incognite  $u$ ,  $v$ ,  $w$

$$\begin{cases} \nabla^2 u + \frac{1}{1-2\nu} \cdot \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) + \frac{X}{G} = 0 \\ \nabla^2 v + \frac{1}{1-2\nu} \cdot \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) + \frac{Y}{G} = 0 \\ \nabla^2 w + \frac{1}{1-2\nu} \cdot \frac{\partial}{\partial z} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) + \frac{Z}{G} = 0 \end{cases}$$

**Problema sostitutivo:** determinare delle funzioni sostitutive che approssimino  $u$ ,  $v$  e  $w$  con un errore accettabile ai fini pratici e siano relativamente facili da calcolare

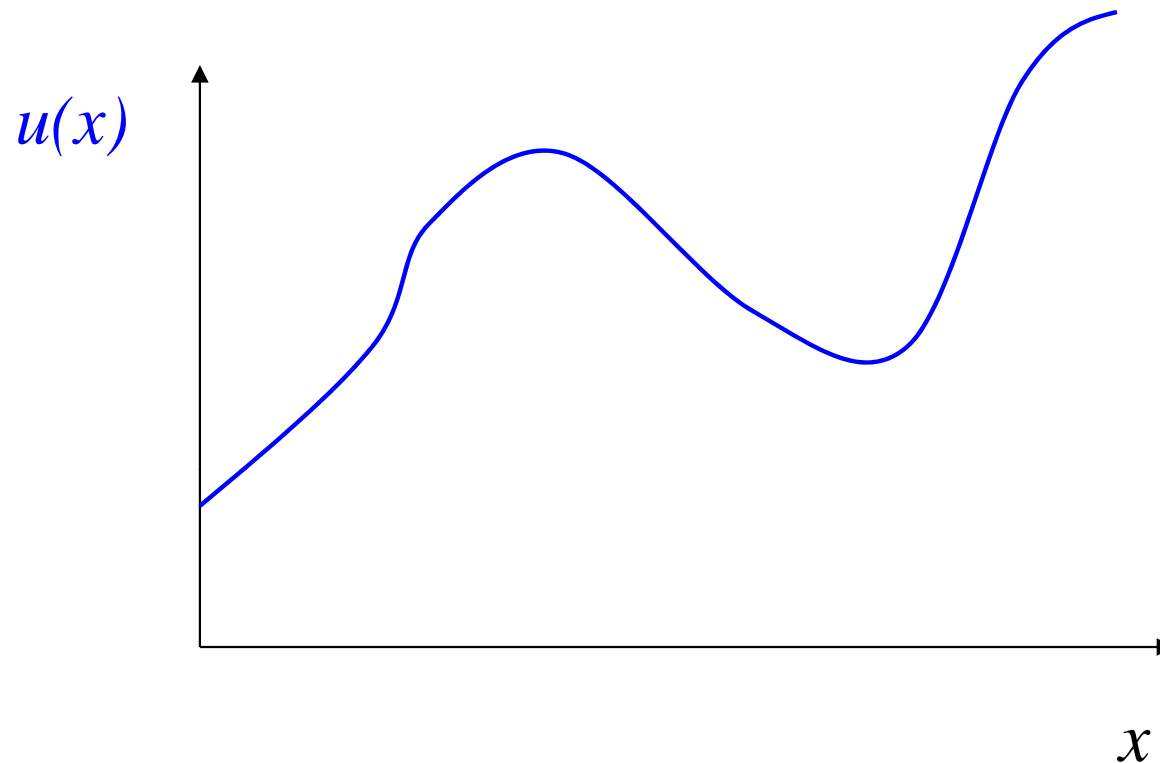


## Esempio di funzione approssimante (problema monodimensionale)





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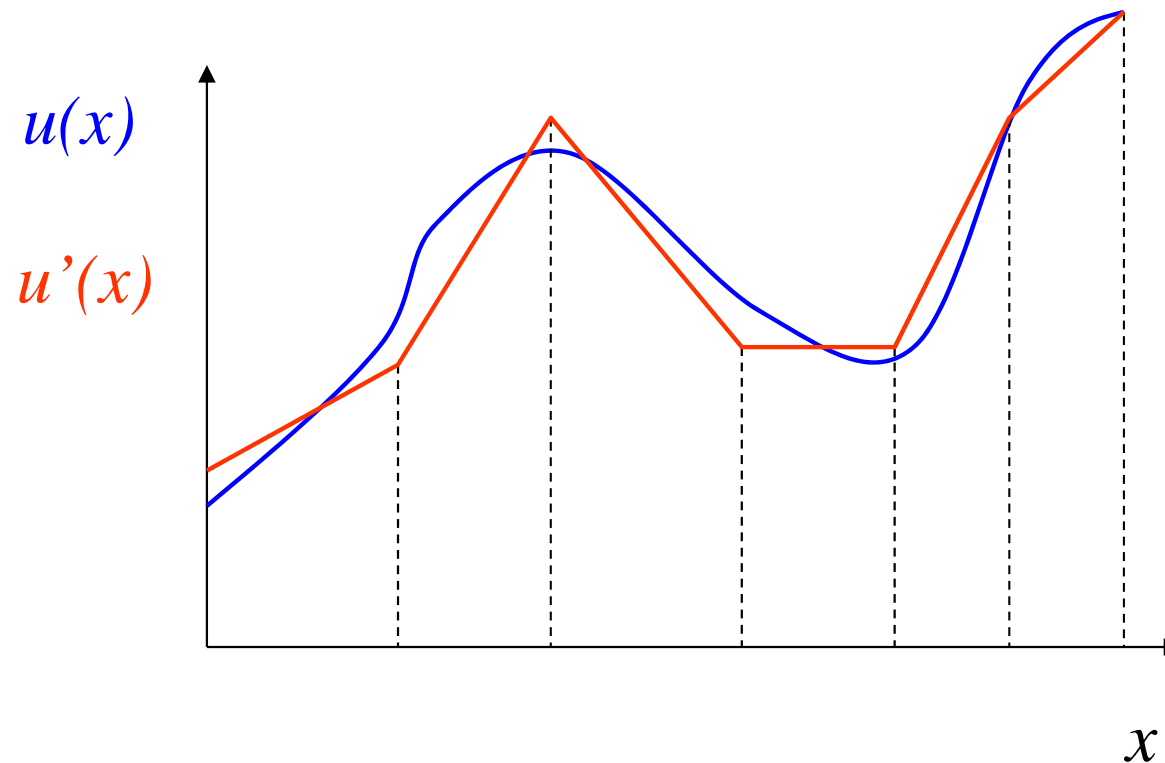


F.ne sostitutiva  $u'(x)$ :

- espressione matematica semplice
- nota ovunque una volta noto il valore di un  $n^\circ$  finito di parametri



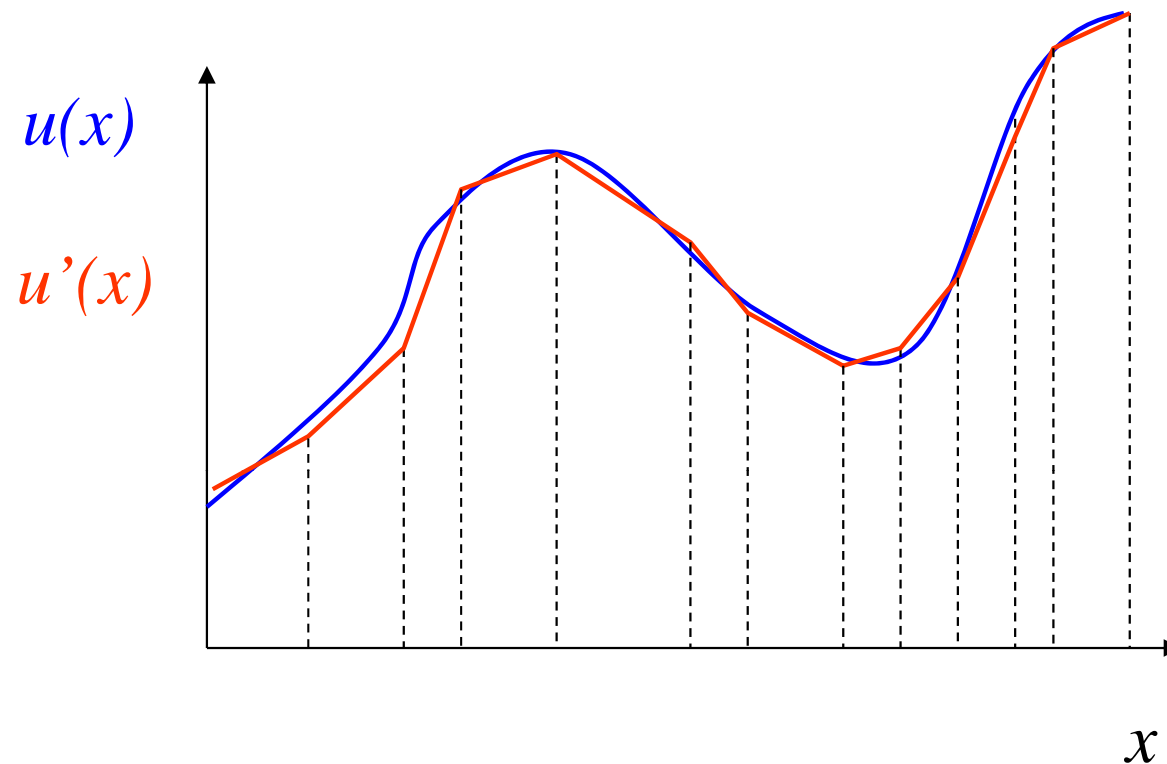
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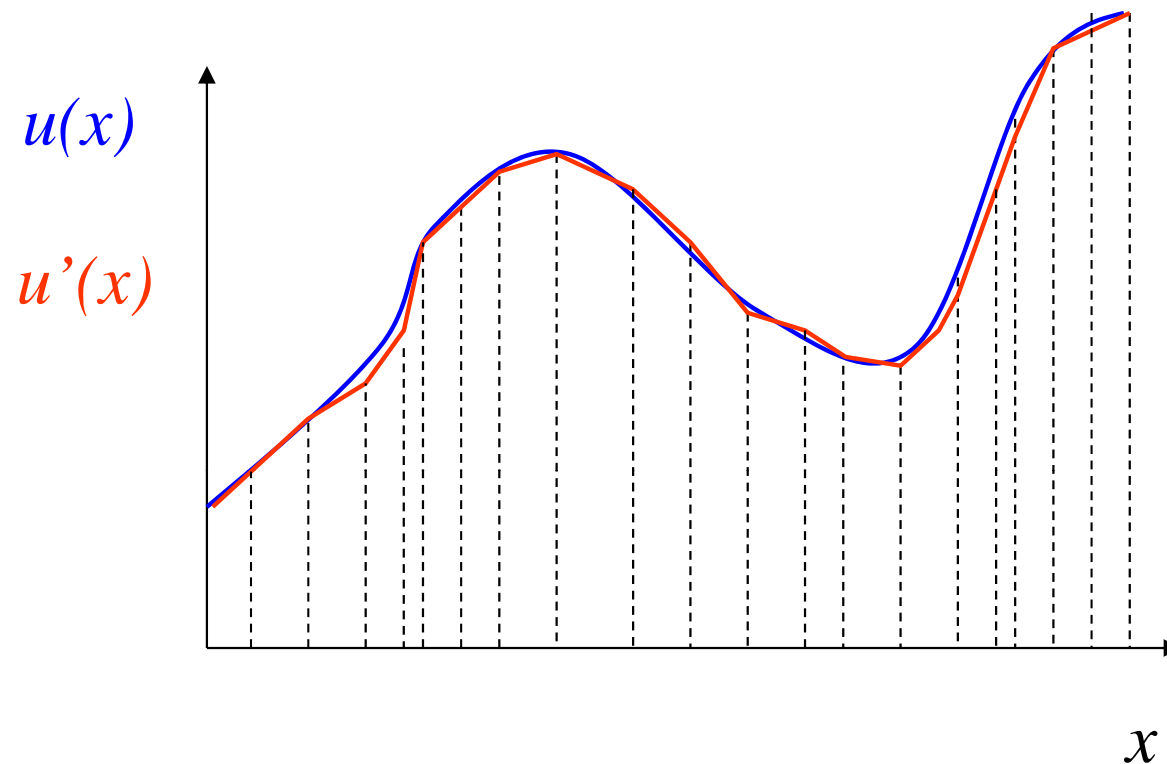
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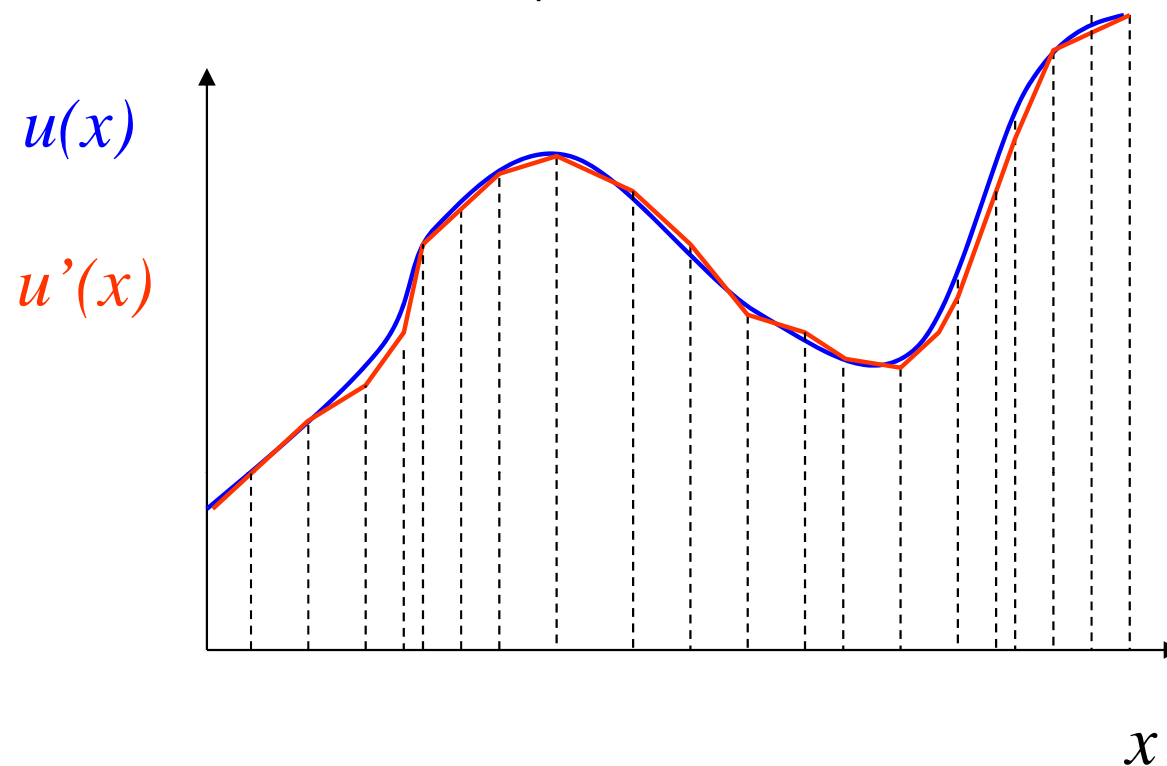
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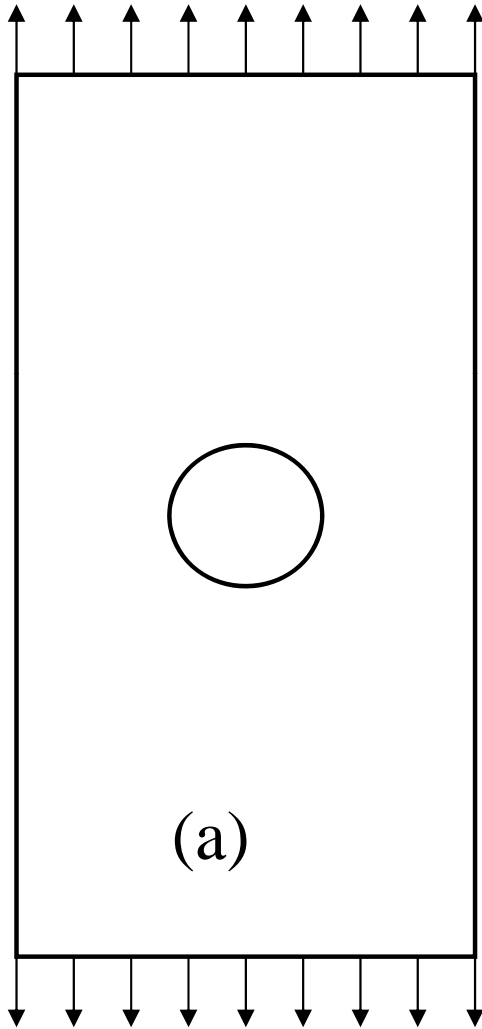
- espressione matematica semplice
- nota ovunque una volta noto il valore di un  $n^\circ$  finito di parametri

Oss.ni:

- necessario assicurare la **convergenza**
- soluzione affetta da **errori**

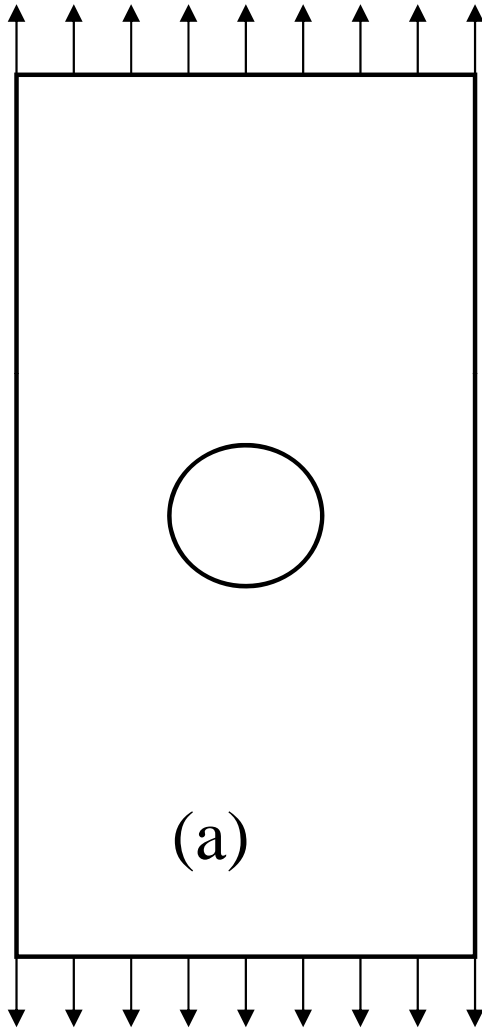


# Discretizzazione

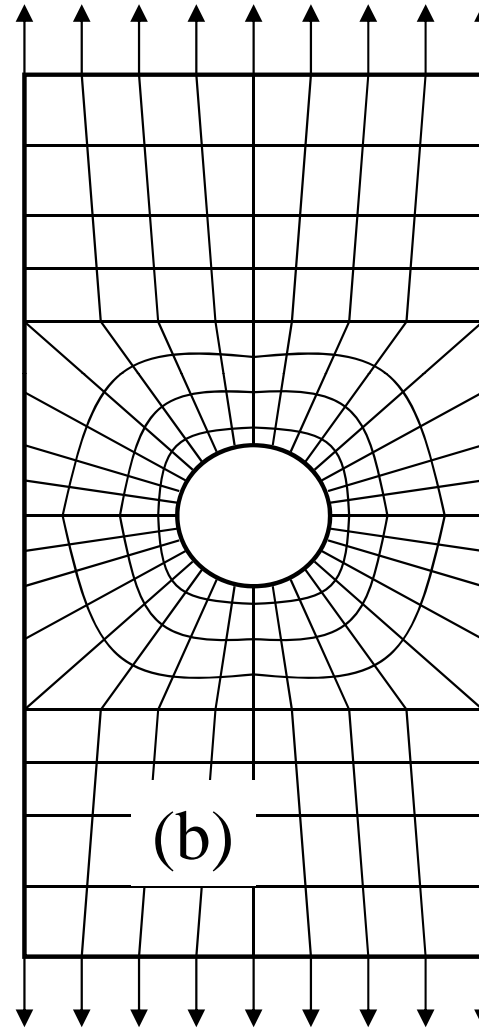


Struttura

# Discretizzazione



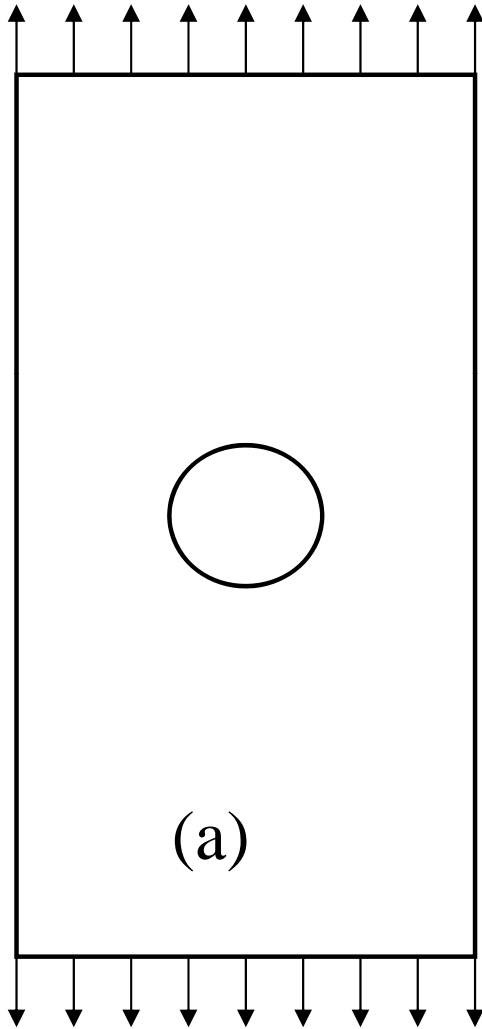
Struttura



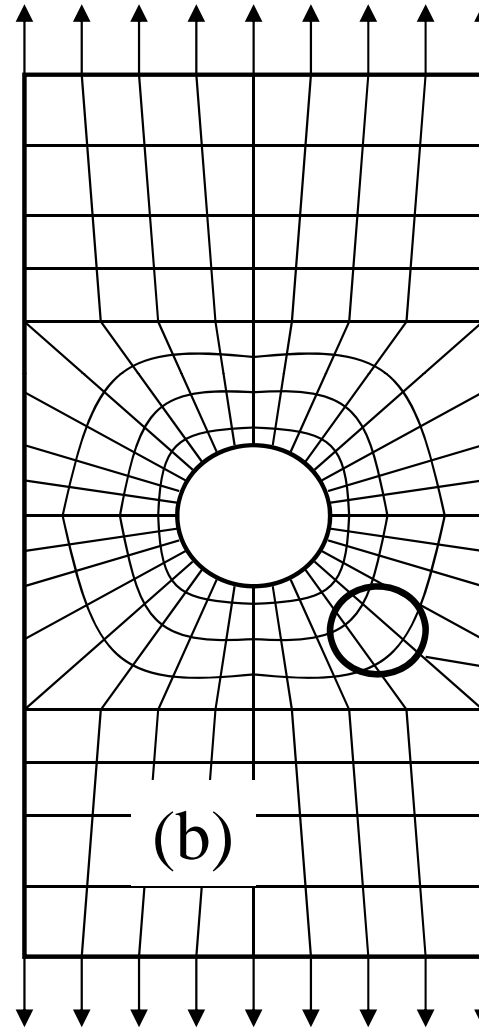
Modello ("mesh")



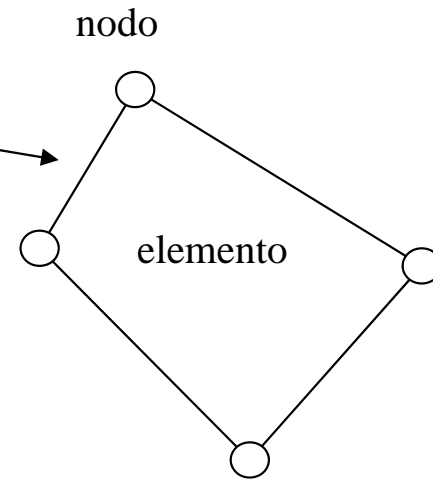
# Discretizzazione



Struttura

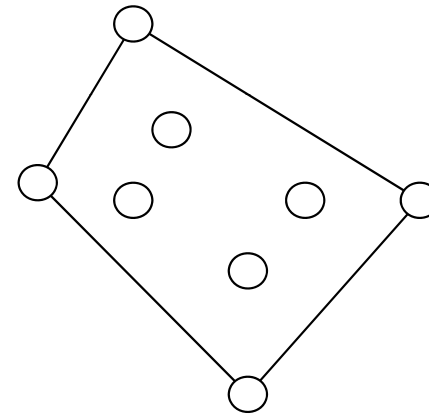
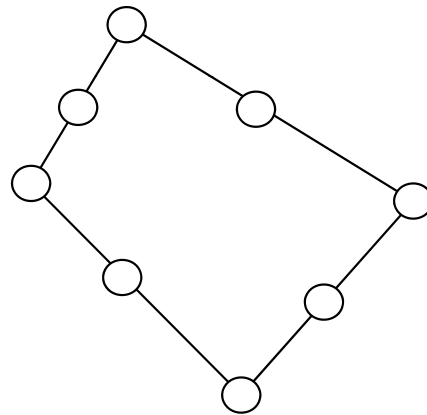
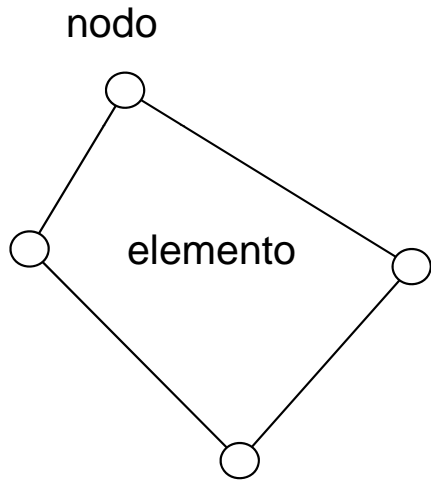


Modello ("mesh")





# Esempi di elementi piani con diverse disposizioni dei nodi



## Nodi ed elementi identificati da un numero univoco

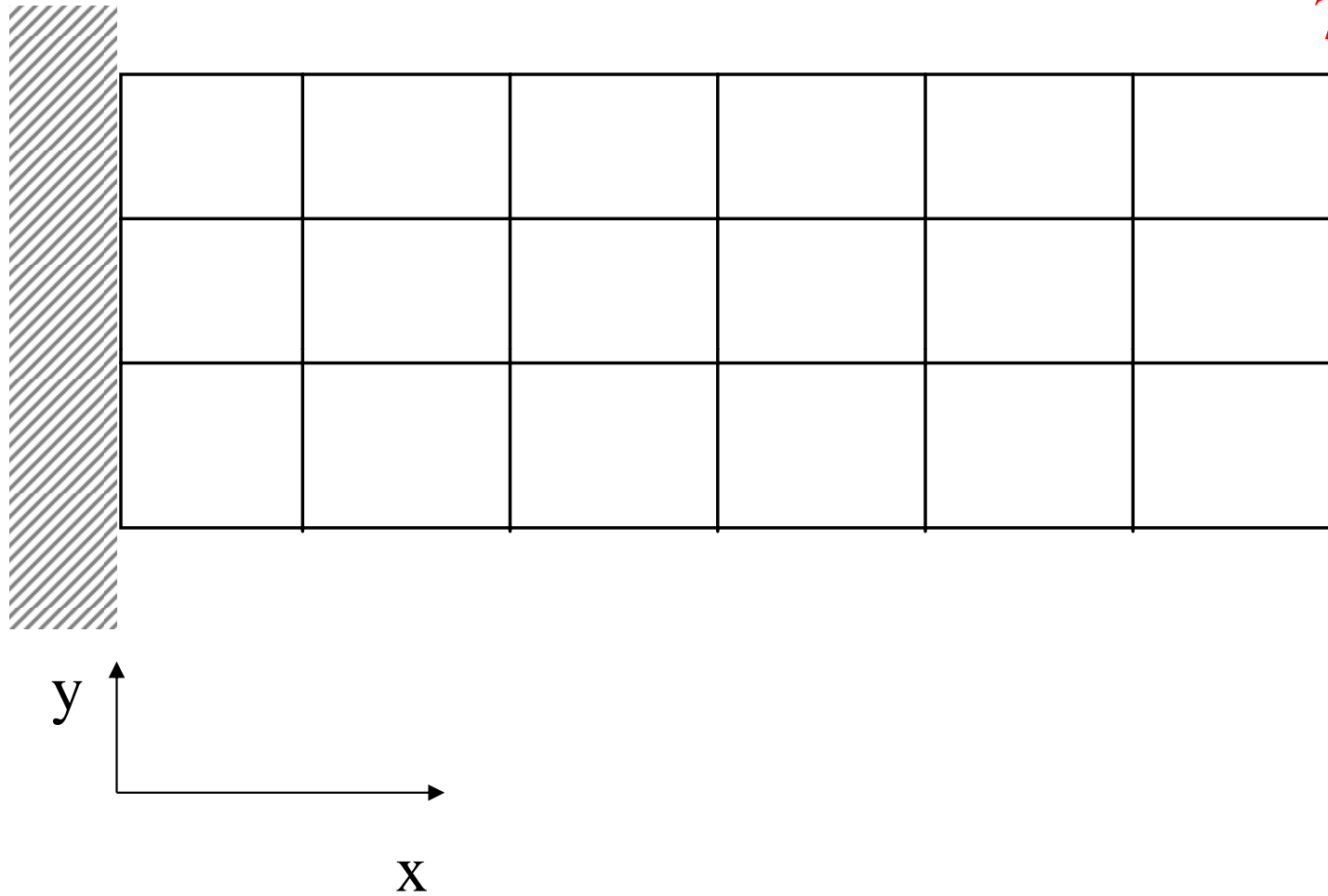
	1	2	3	4	5	6	7
	1	2	3	4	5	6	
8	9	10	11	12	13	14	
7	8	9	10	11	12		
15	16	17	18	19	20	21	
13	14	15	16	17	18		
22	23	24	25	26	27	28	

$i = n^{\circ}$  di nodo

$i = n^{\circ}$  di elemento

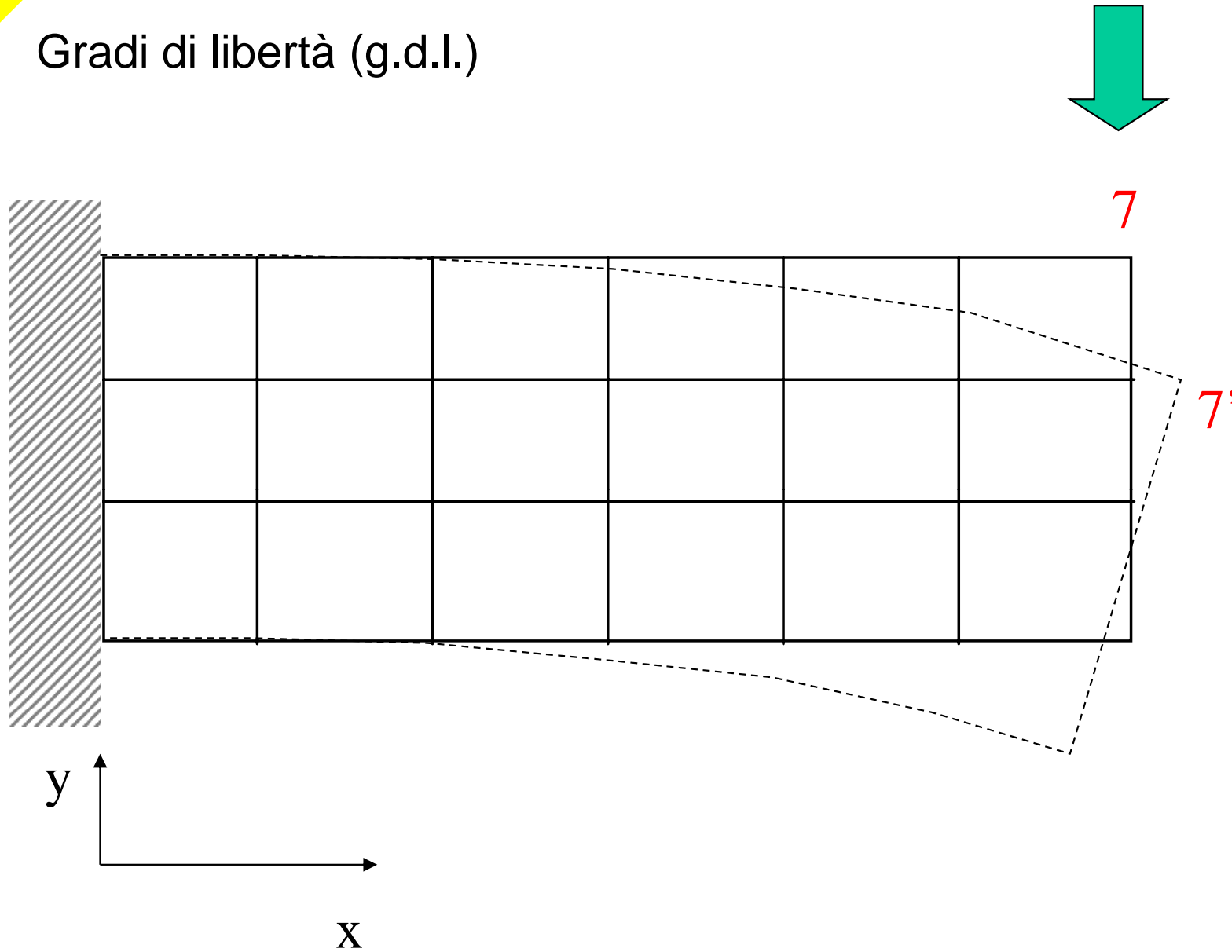


# Gradi di libertà (g.d.l.)





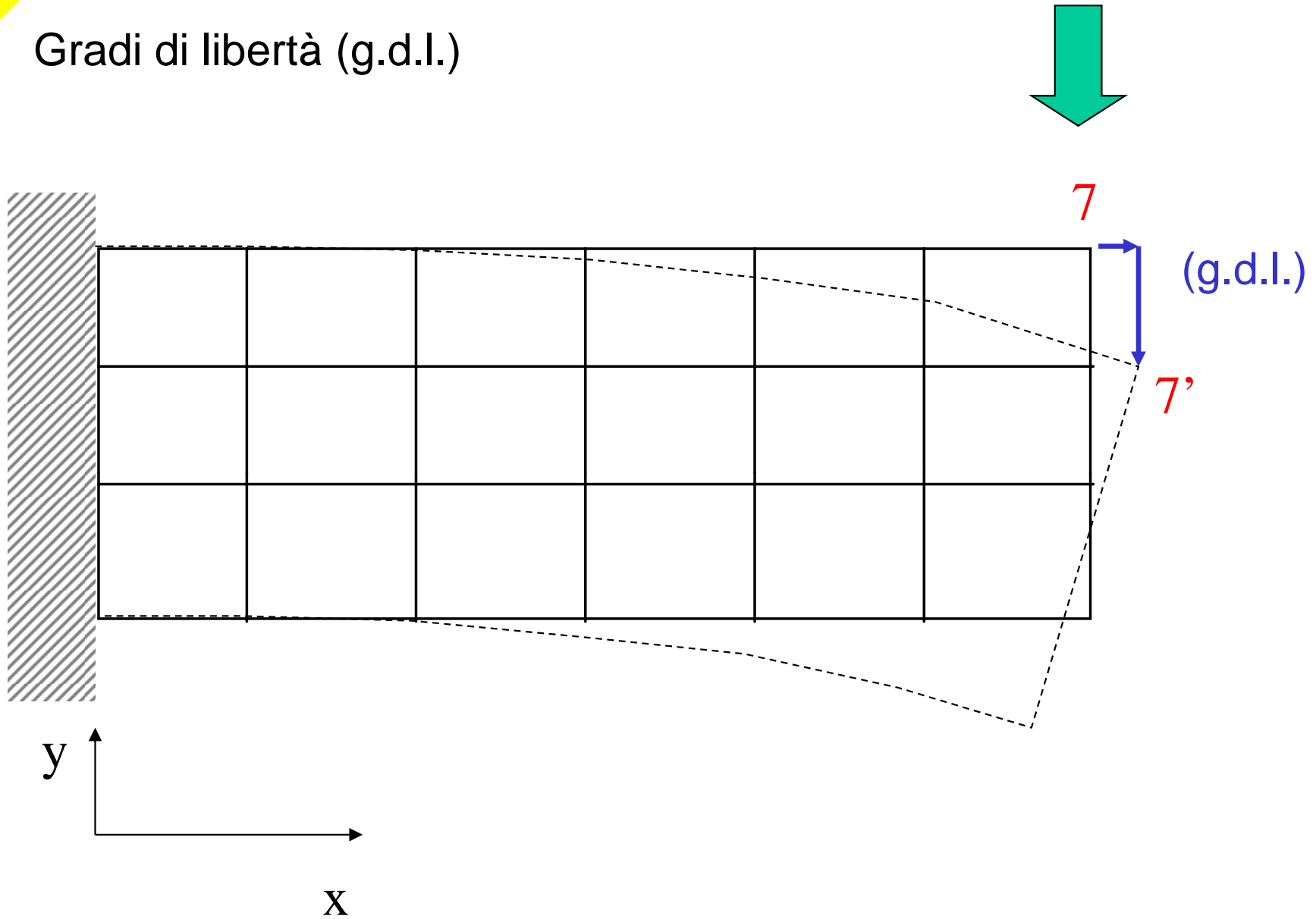
# Gradi di libertà (g.d.l.)



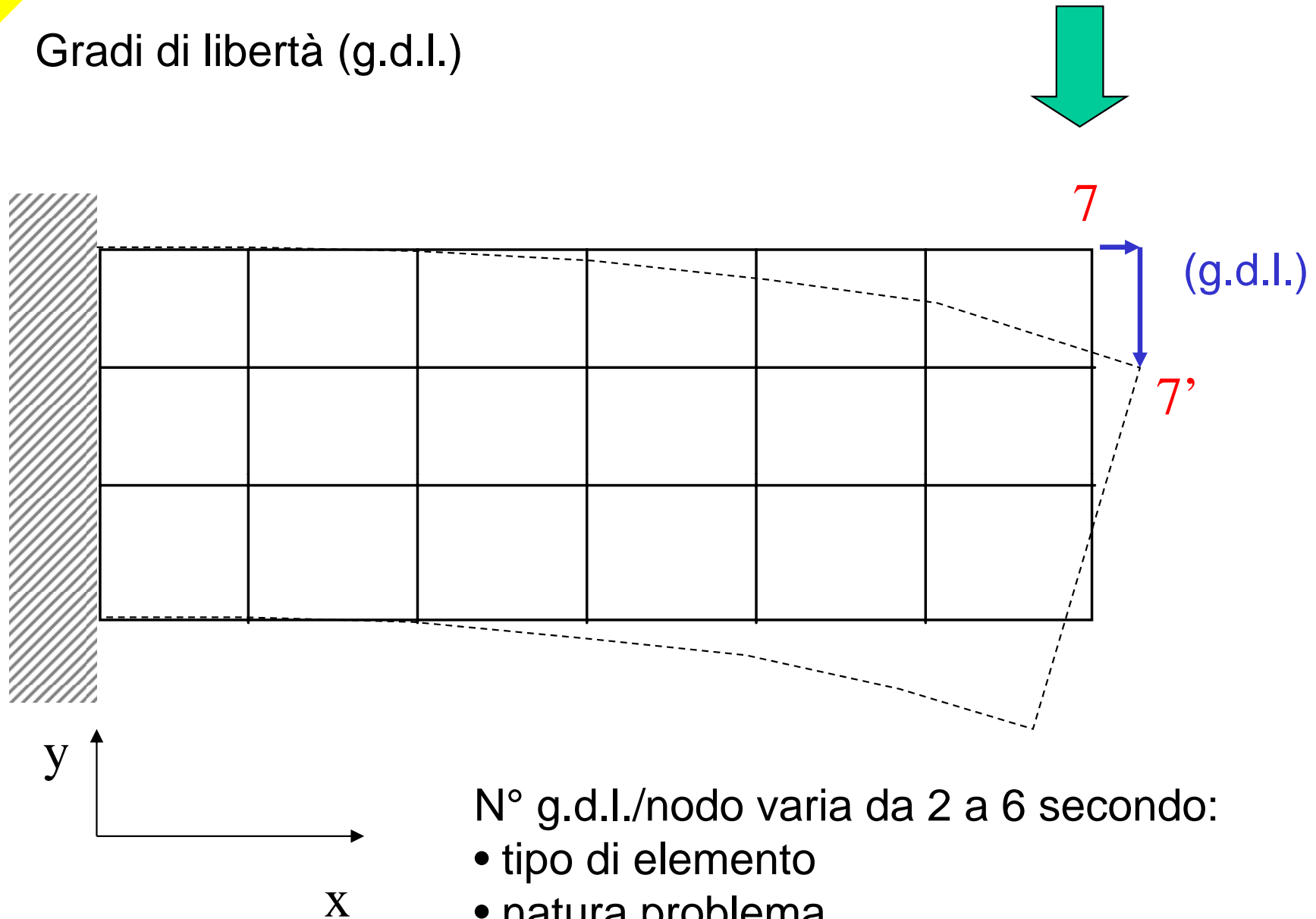




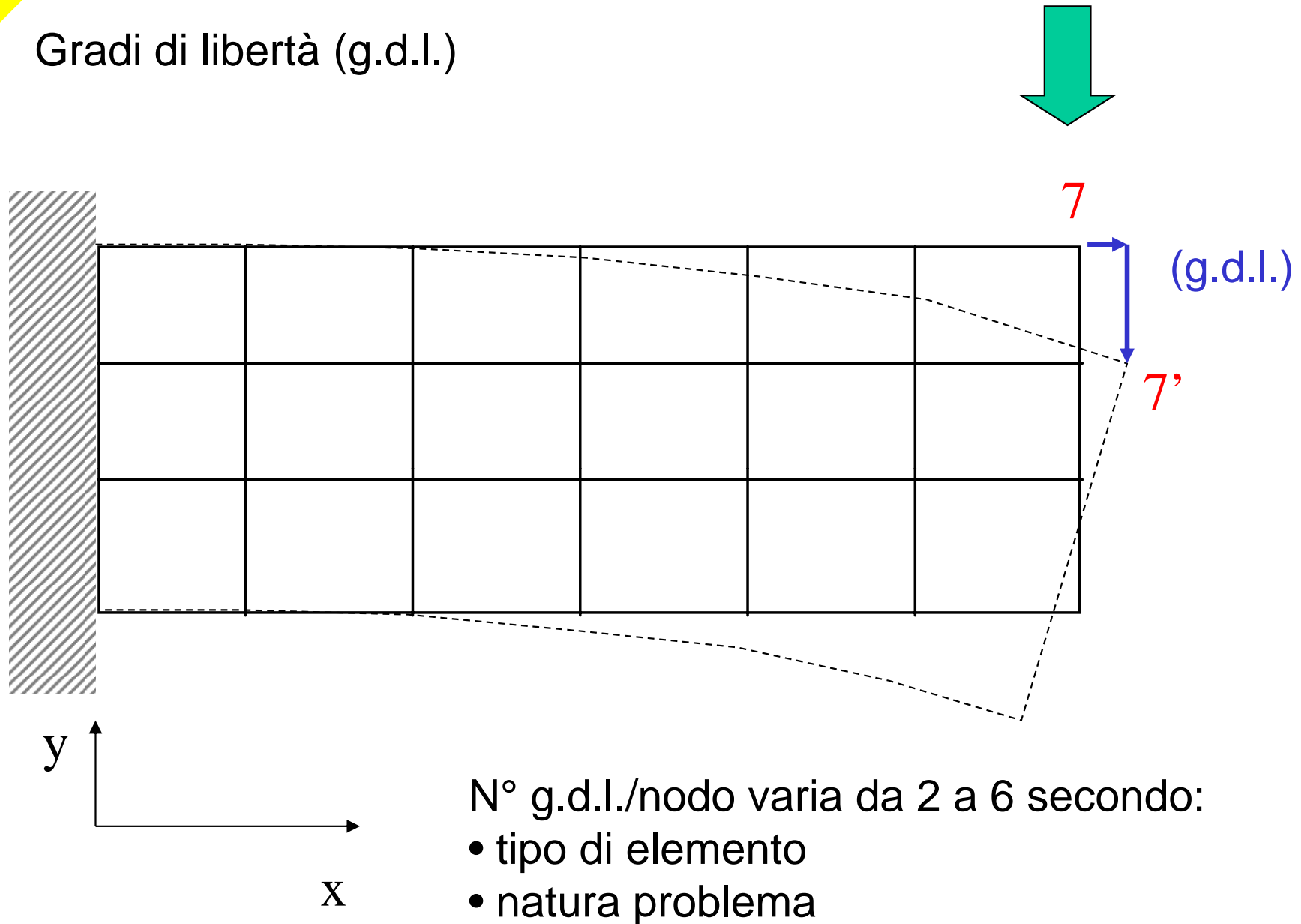
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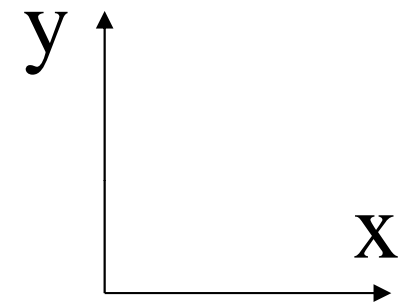
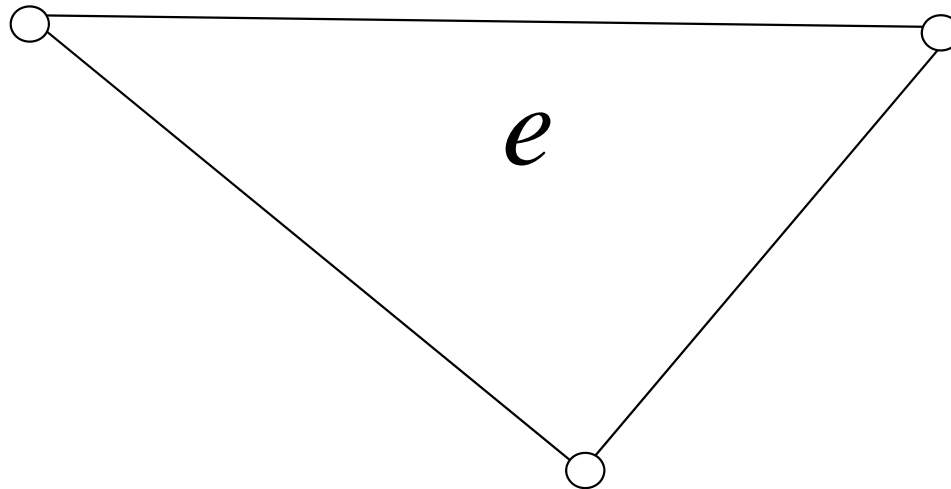


$$N^{\circ} \text{ totale g.d.l.} = N^{\circ} \text{ g.d.l./nodo} * N^{\circ} \text{ nodi}$$



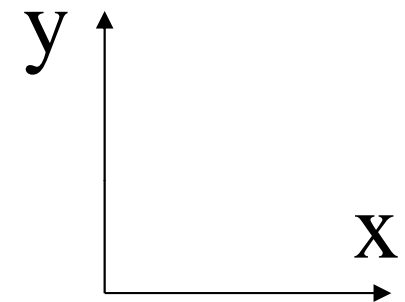
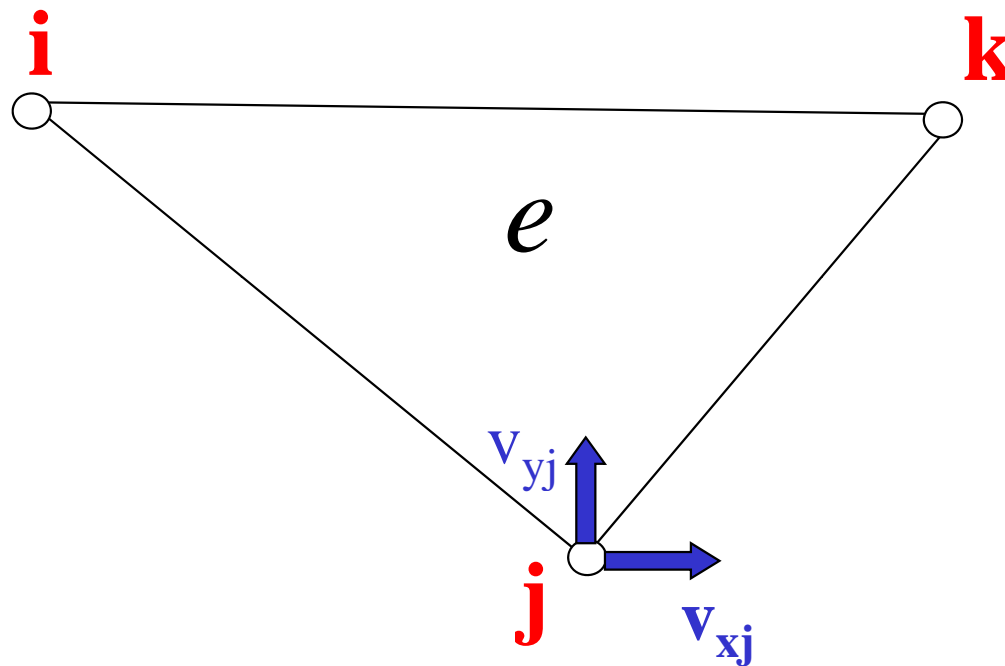
# Studio del comportamento meccanico del singolo elemento

## Elemento piano per problemi 2D



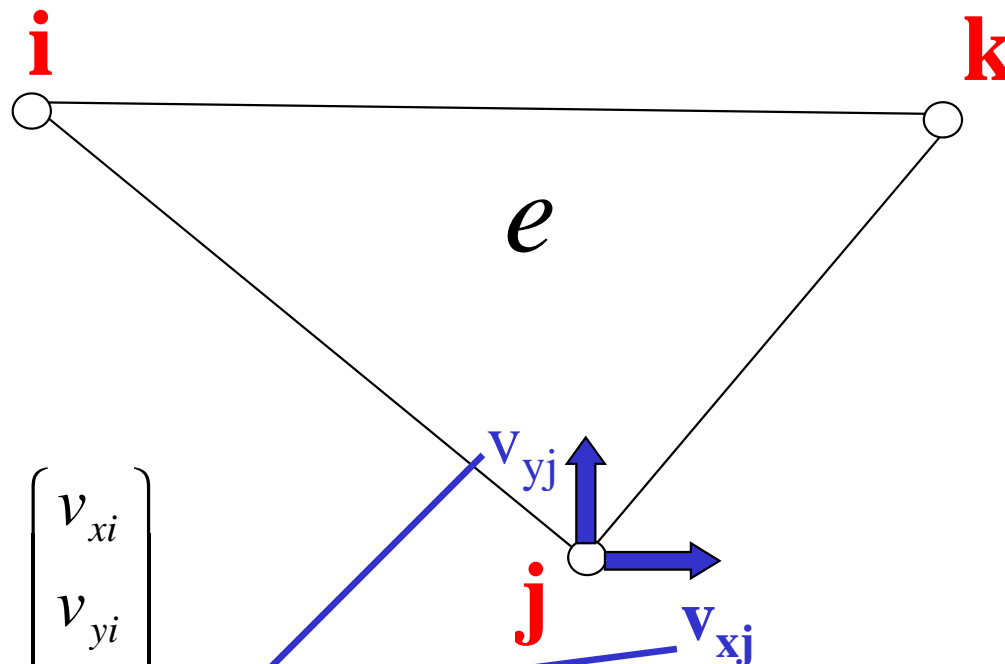
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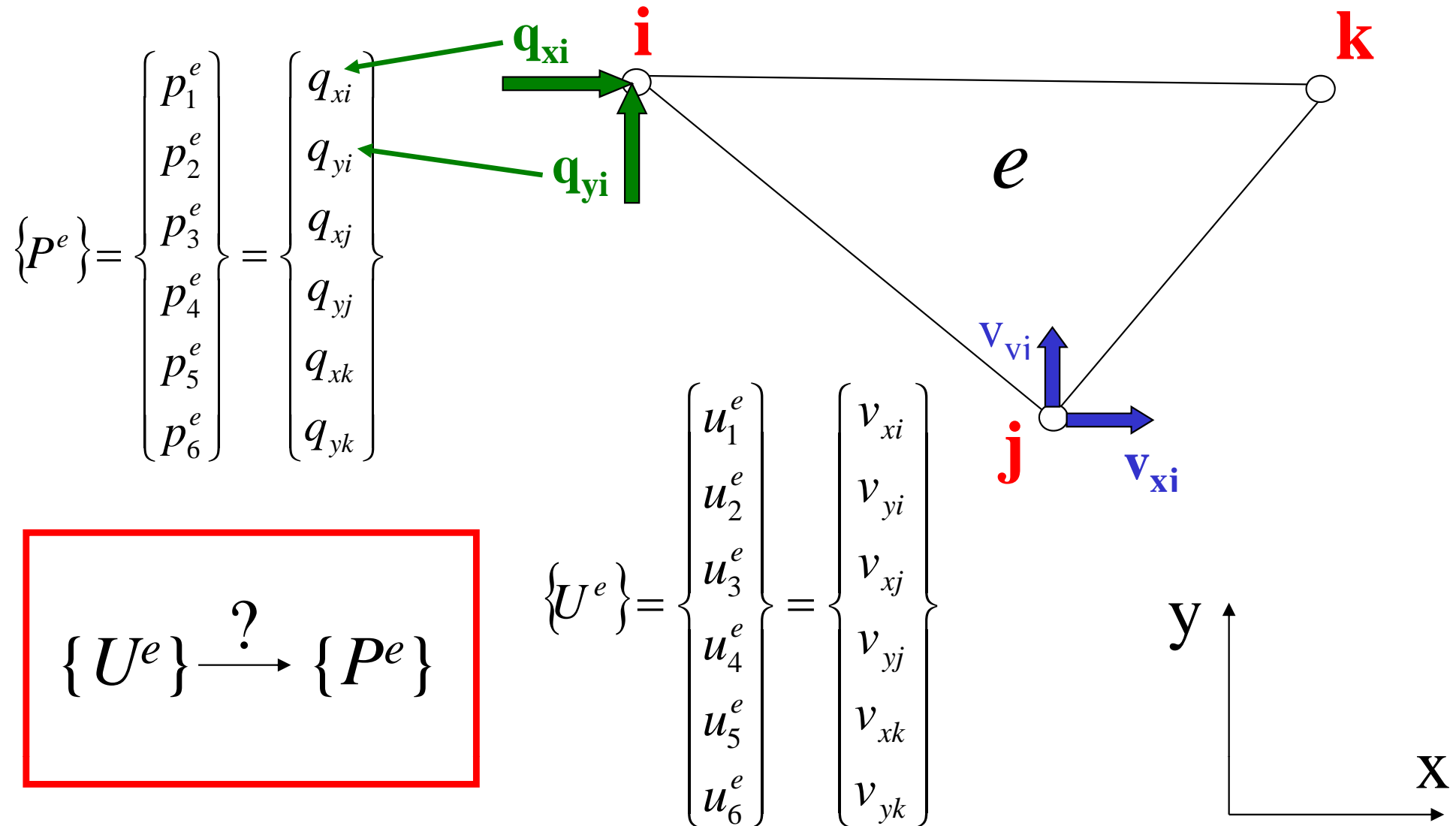


$$\{U^e\} = \begin{Bmatrix} u_1^e \\ u_2^e \\ u_3^e \\ u_4^e \\ u_5^e \\ u_6^e \end{Bmatrix} = \begin{Bmatrix} v_{xi} \\ v_{yi} \\ v_{xj} \\ v_{yj} \\ v_{xk} \\ v_{yk} \end{Bmatrix}$$

(6 x 1)

## Studio del comportamento meccanico del singolo elemento

## Elemento piano per problemi 2D





Studio condotto in campo lineare:

$$\left\{ P^e \right\} = \left[ K^e \right] \cdot \left\{ U^e \right\}$$

$6 \times 1$        $6 \times 6$        $6 \times 1$

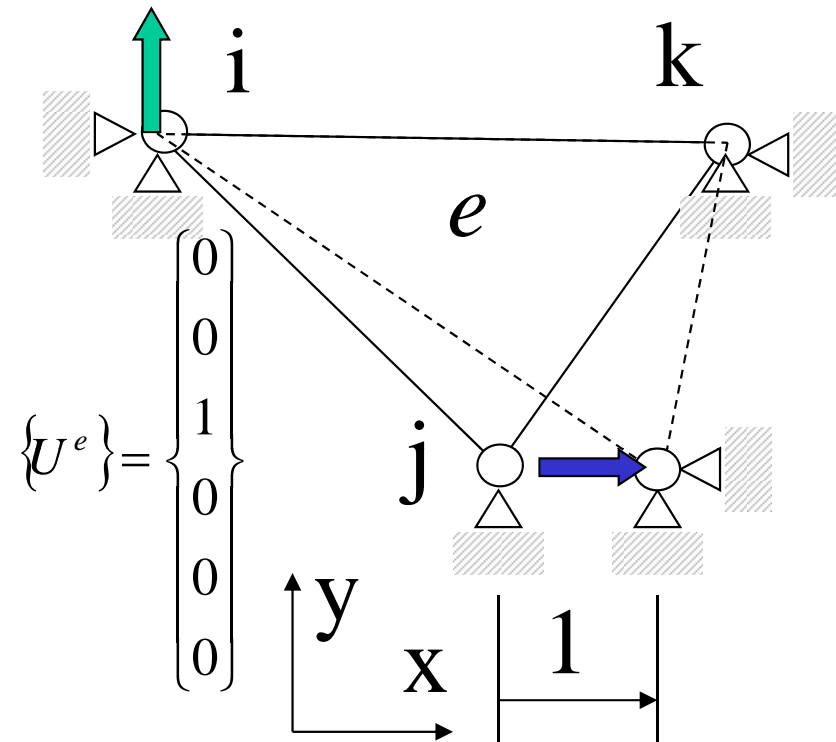
**Matrice di rigidezza** dell'elemento



Significato fisico dei termini della matrice di rigidezza,  $k_{ij}$

“Cedimento” vincolare:

$$\begin{Bmatrix} p_1^e \\ p_2^e \\ p_3^e \\ p_4^e \\ p_5^e \\ p_6^e \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} & k_{15} & k_{16} \\ k_{21} & k_{22} & k_{23} & k_{24} & k_{25} & k_{26} \\ k_{31} & k_{32} & k_{33} & k_{34} & k_{35} & k_{36} \\ k_{41} & k_{42} & k_{43} & k_{44} & k_{45} & k_{46} \\ k_{51} & k_{52} & k_{53} & k_{54} & k_{55} & k_{56} \\ k_{61} & k_{62} & k_{63} & k_{64} & k_{65} & k_{66} \end{bmatrix} \cdot \begin{Bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$



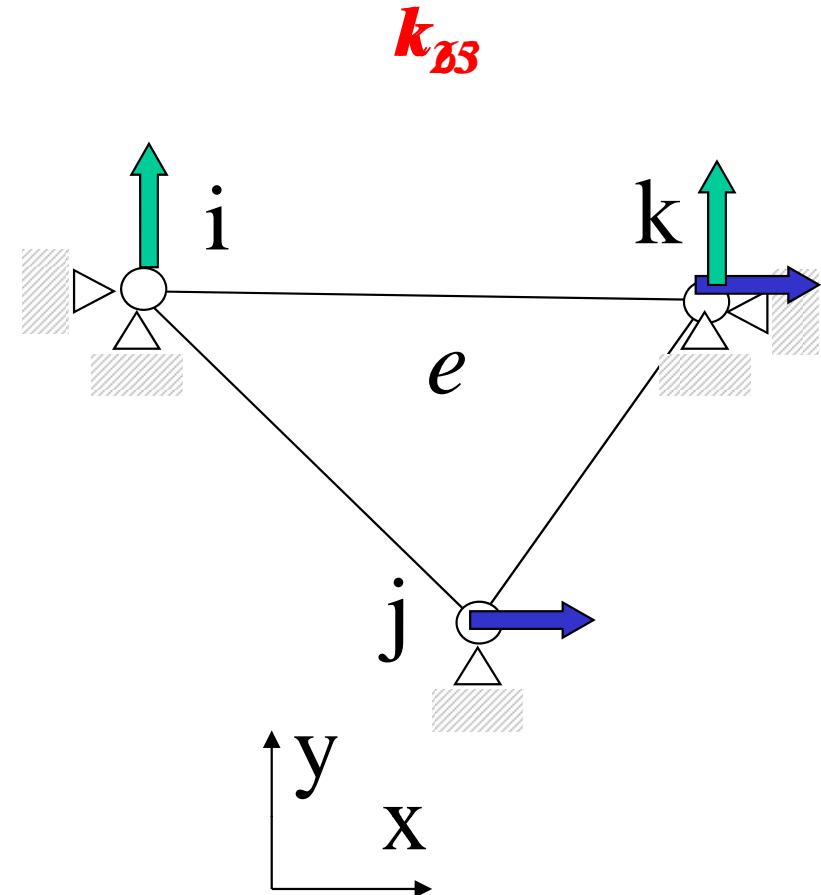
$$p_2^e = k_{21} \cdot 0 + k_{22} \cdot 0 + k_{23} \cdot 1 + \dots + k_{26} \cdot 0 = k_{23}$$

$$p_3^e = k_{33}; \quad p_4^e = k_{43} \dots$$

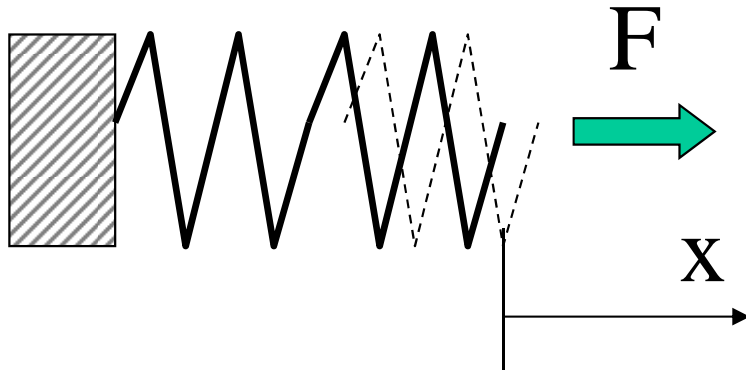
Il termine  $k_{m,n}$  di  $[K^e]$  è pari alla reazione vincolare presente secondo il grado di libertà "m" ( $m=1,..6$ ), se si applica un sistema di spostamenti nodali in cui tutte le componenti sono nulle tranne la "n-esima" che assume valore pari ad 1

$$p_m^e = \sum_n k_{m,n}^e u_n^e$$

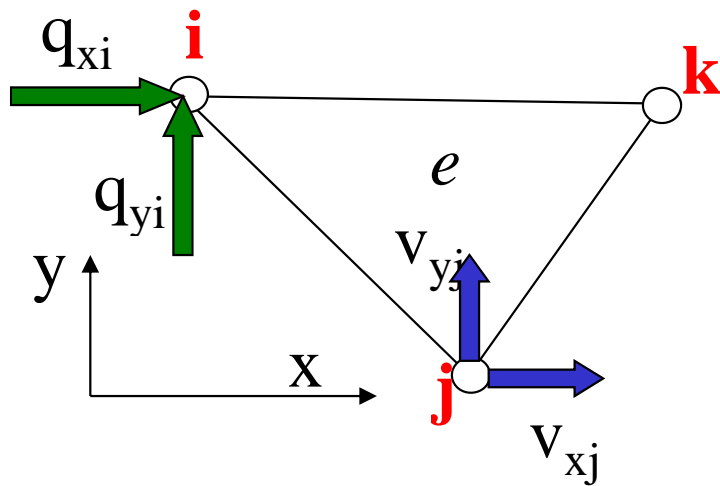
"peso" di  $u_n$  nel contribuire a  $p_m$



Elemento = molla “multidimensionale



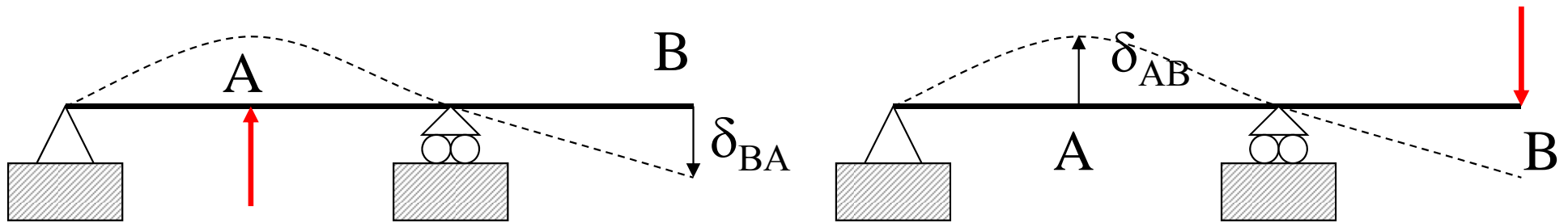
$$F = k x$$



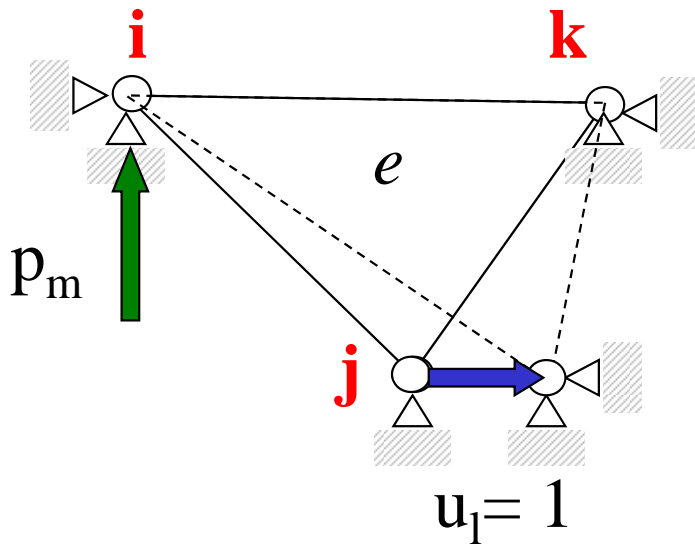
$$\{P^e\} = [K^e] \cdot \{U^e\}$$



## Teorema di reciprocità



$$\delta_{AB} = \delta_{BA}$$



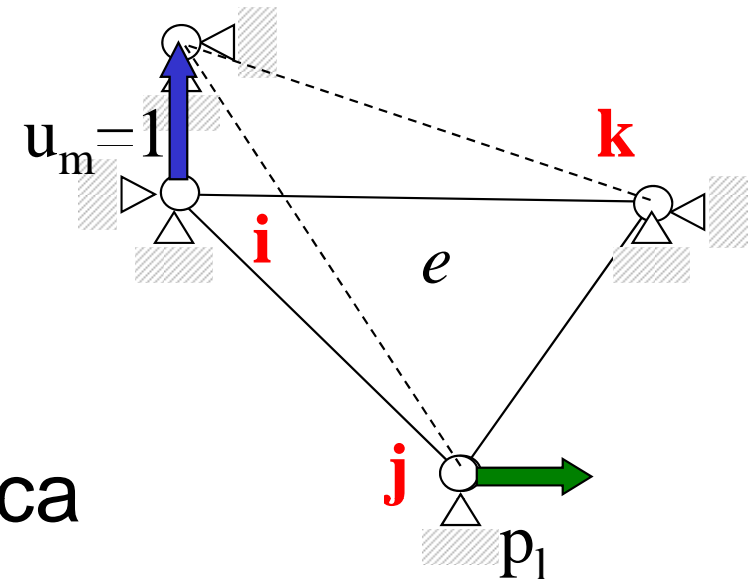
$$p_m^e = p_l^e$$



$$k_{ml} = k_{lm}$$

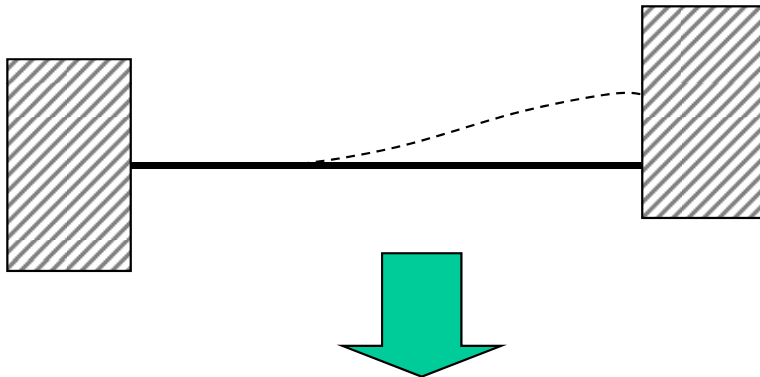


$[K^e]$  simmetrica



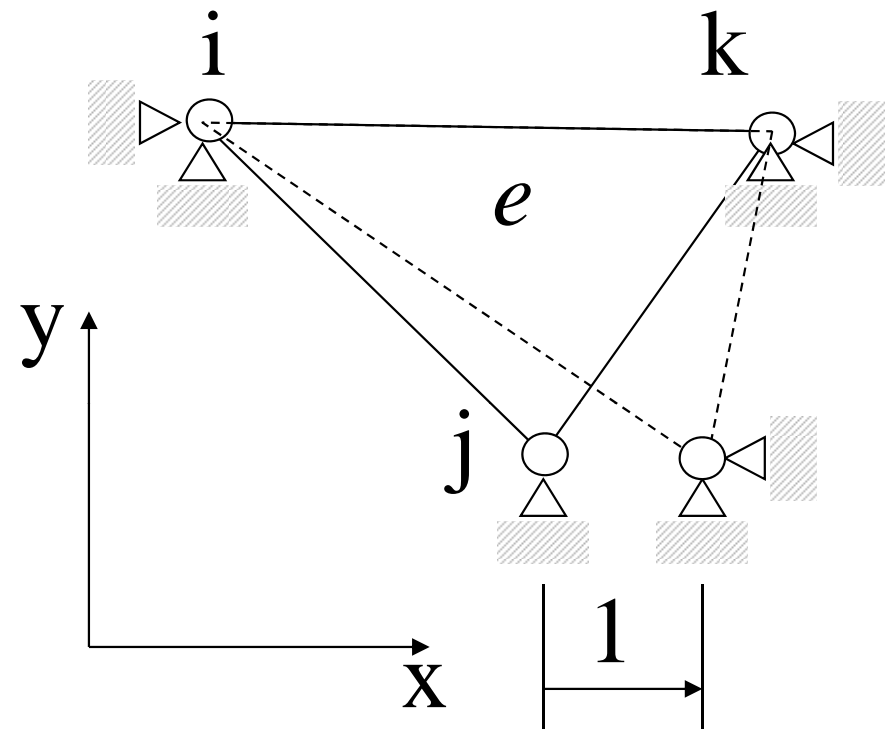
Valutazione di  $[K^e]$ 

In casi semplici è possibile calcolare le reazioni vincolari in presenza di “cedimenti vincolari” dei nodi (Es. elementi trave)



si ottengono immediatamente le

$$k_{m,n}^e$$



In generale, questa procedura non è praticabile per un elemento di forma generica

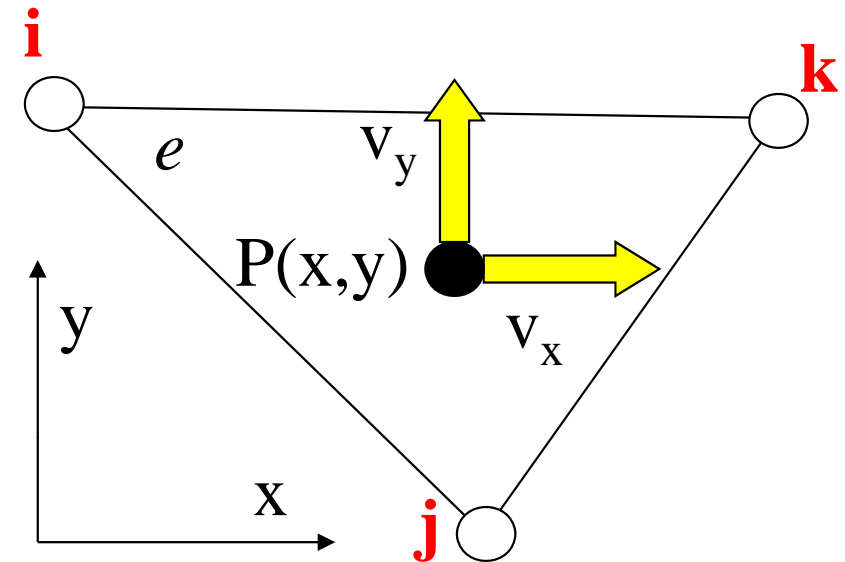
## Spostamenti nei punti interni all'elemento

$$\{v(x, y)\} = \begin{Bmatrix} v_x(x, y) \\ v_y(x, y) \end{Bmatrix} = [N^e(x, y)] \cdot \{U^e\}$$

$2 \times 1$                    $2 \times 1$                    $2 \times 6$                    $6 \times 1$

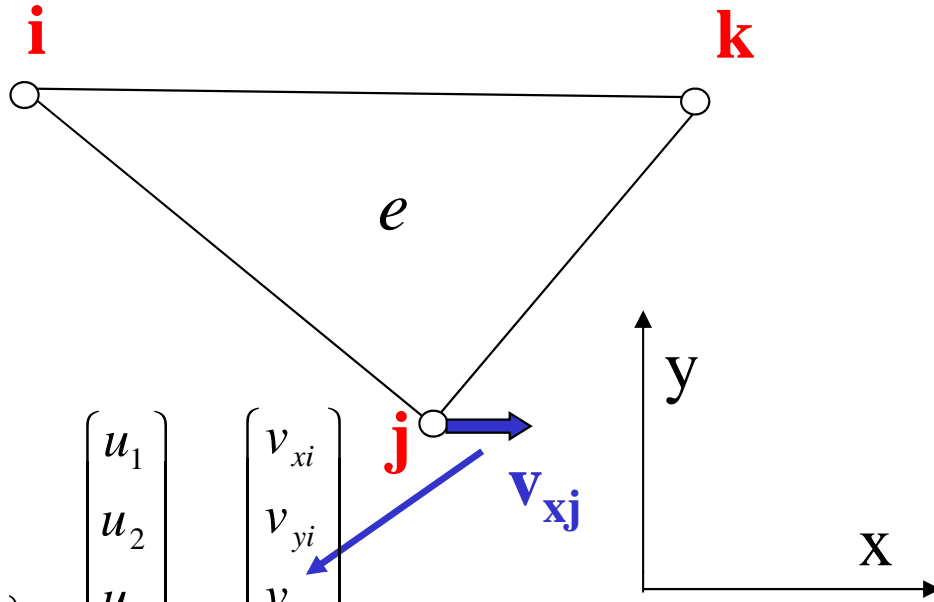
**F.ni di forma (“shape functions”)**

$$v_r = \sum_{l=1}^6 N_{rl}^e(x, y) \cdot u_l$$

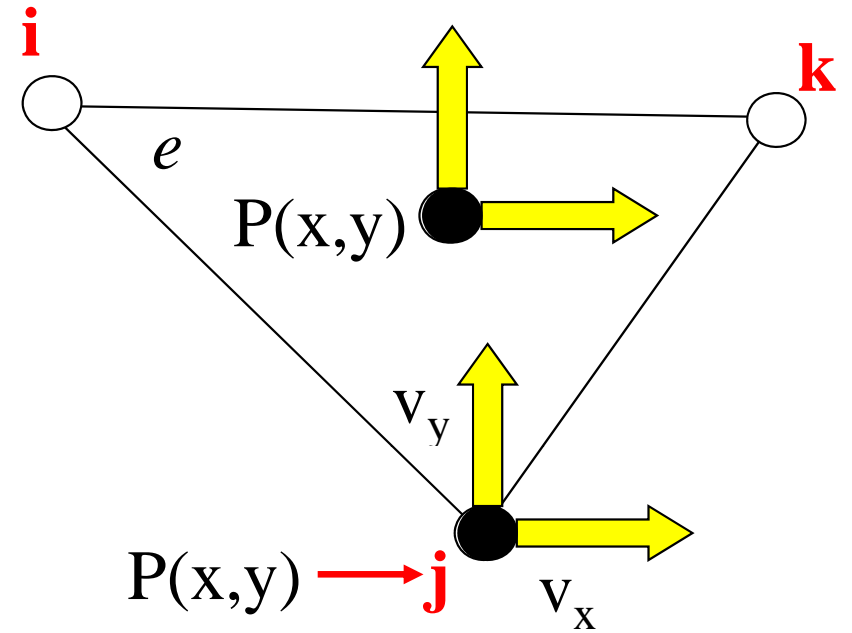


Ogni f.ne di forma rappresenta il “peso” (dipendente dalla posizione di P) che ciascuna componente di spostamento nodale ha nel determinare lo spostamento di P

Pb: - che forma matematica dare alle  $N^e(x, y)$  ?  
 - come determinare le  $N^e(x, y)$  ?



$$\{U^e\} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix} = \begin{Bmatrix} v_{xi} \\ v_{yi} \\ v_{xj} \\ v_{yj} \\ v_{xk} \\ v_{yk} \end{Bmatrix}$$



$$v_1(x_j, y_j) = v_x(x_j, y_j) = \sum_{l=1}^6 N_{1l}^e(x_j, y_j) \cdot u_l =$$

$$= N_{11}^e(x_j, y_j) \cdot u_1 + N_{12}^e(x_j, y_j) \cdot u_2 + \dots = u_3$$

$$N_{1l}^e(x_j, y_j) = \begin{cases} 0 & \text{se } l \neq 3 \\ 1 & \text{se } l = 3 \end{cases}$$

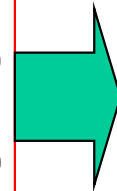


$$v_1(x_j, y_j) = \sum_{l=1}^6 N_{1l}^e(x_j, y_j) \cdot u_l = N_{11}^e(x_i, y_i) \cdot u_1 + N_{12}^e(x_i, y_i) \cdot u_2 + \dots$$

$$\begin{cases} N_{11}^e(x_i, y_i) = 1 & N_{14}^e(x_i, y_i) = 0 \\ N_{12}^e(x_i, y_i) = 0 & N_{15}^e(x_i, y_i) = 0 \\ N_{13}^e(x_i, y_i) = 0 & N_{16}^e(x_i, y_i) = 0 \end{cases}$$

$$\begin{cases} N_{11}^e(x_j, y_j) = 0 & N_{14}^e(x_j, y_j) = 0 \\ N_{12}^e(x_j, y_j) = 0 & N_{15}^e(x_j, y_j) = 0 \\ N_{13}^e(x_j, y_j) = 1 & N_{16}^e(x_j, y_j) = 0 \end{cases}$$

$$\begin{cases} N_{11}^e(x_k, y_k) = 0 & N_{14}^e(x_k, y_k) = 0 \\ N_{12}^e(x_k, y_k) = 0 & N_{15}^e(x_k, y_k) = 1 \\ N_{13}^e(x_k, y_k) = 0 & N_{16}^e(x_k, y_k) = 0 \end{cases}$$



$$\begin{cases} N_{11}(x_i, y_i) = 1 \\ N_{11}(x_j, y_j) = 0 \\ N_{11}(x_k, y_k) = 0 \end{cases} \quad \begin{cases} N_{12}(x_i, y_i) = 0 \\ N_{12}(x_j, y_j) = 0 \\ N_{12}(x_k, y_k) = 0 \end{cases}$$

$$\begin{cases} N_{13}(x_i, y_i) = 0 \\ N_{13}(x_j, y_j) = 1 \\ N_{13}(x_k, y_k) = 0 \end{cases} \quad \begin{cases} N_{14}(x_i, y_i) = 0 \\ N_{14}(x_j, y_j) = 0 \\ N_{14}(x_k, y_k) = 0 \end{cases}$$

$$\begin{cases} N_{15}(x_i, y_i) = 0 \\ N_{15}(x_j, y_j) = 0 \\ N_{15}(x_k, y_k) = 1 \end{cases} \quad \begin{cases} N_{16}(x_i, y_i) = 0 \\ N_{16}(x_j, y_j) = 0 \\ N_{16}(x_k, y_k) = 0 \end{cases}$$



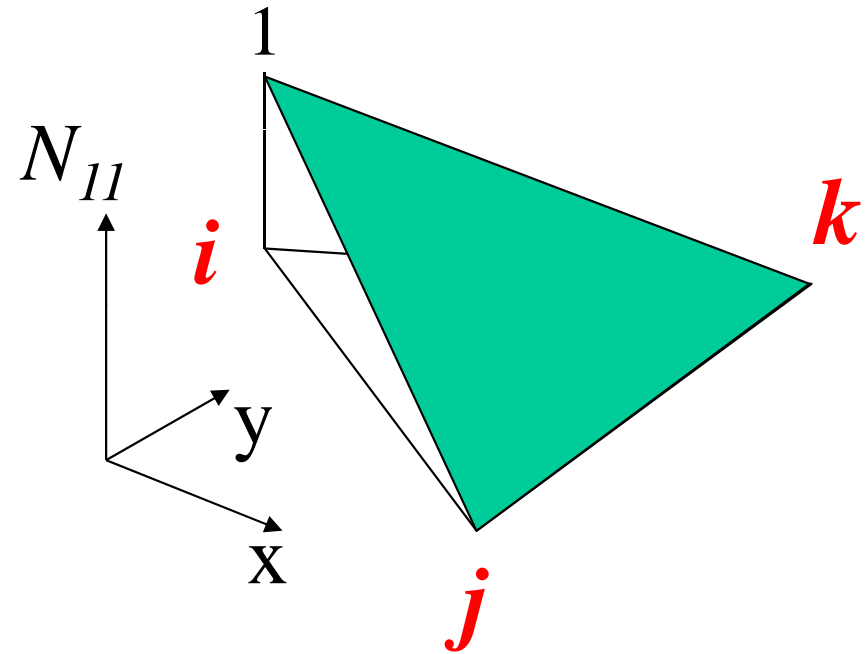


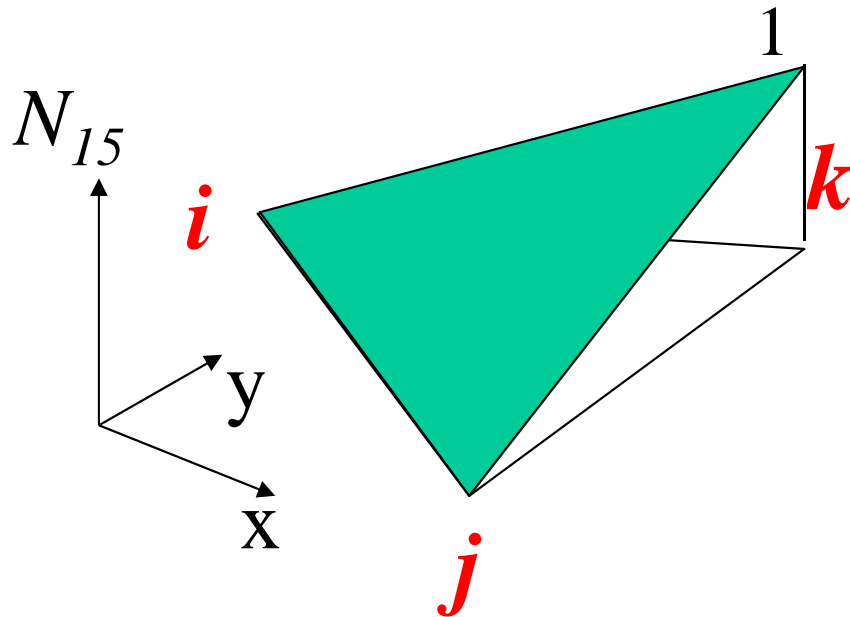
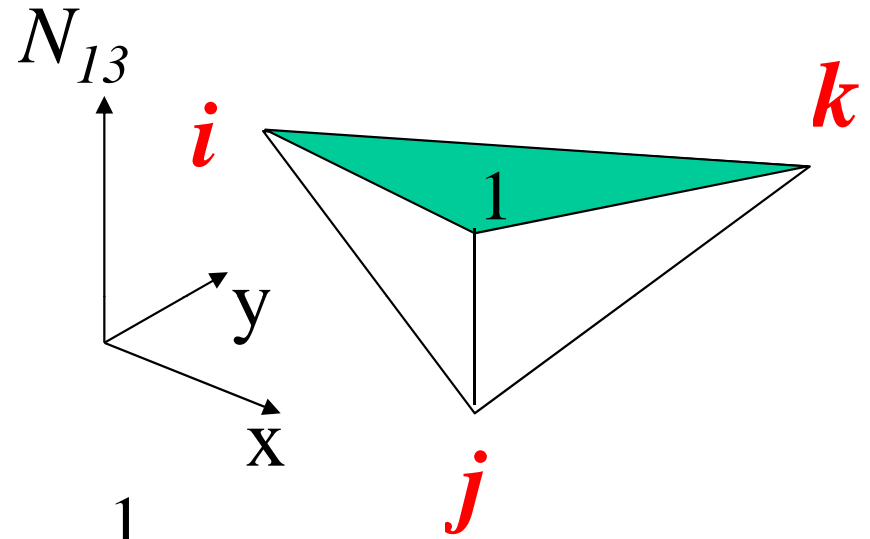
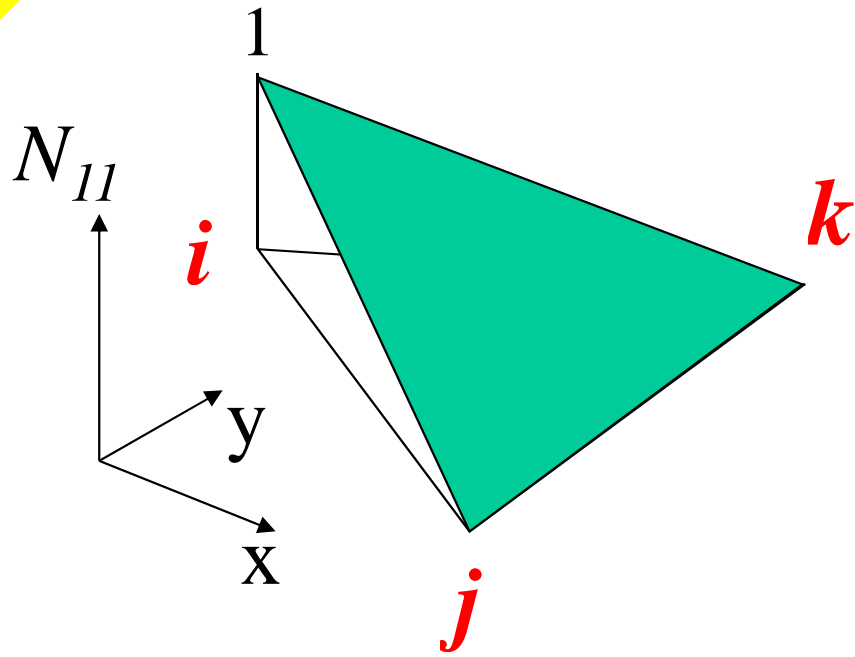
$$\begin{cases} N_{11}(x_i, y_i) = 1 \\ N_{11}(x_j, y_j) = 0 \\ N_{11}(x_k, y_k) = 0 \end{cases}$$

$$N_{lm}^e(x, y) = A_{lm} + B_{lm} \cdot x + C_{lm} \cdot y$$

$$\begin{cases} A_{11} + B_{11}x_i + C_{11}y_i = 1 \\ A_{11} + B_{11}x_j + C_{11}y_j = 0 \\ A_{11} + B_{11}x_k + C_{11}y_k = 0 \end{cases}$$

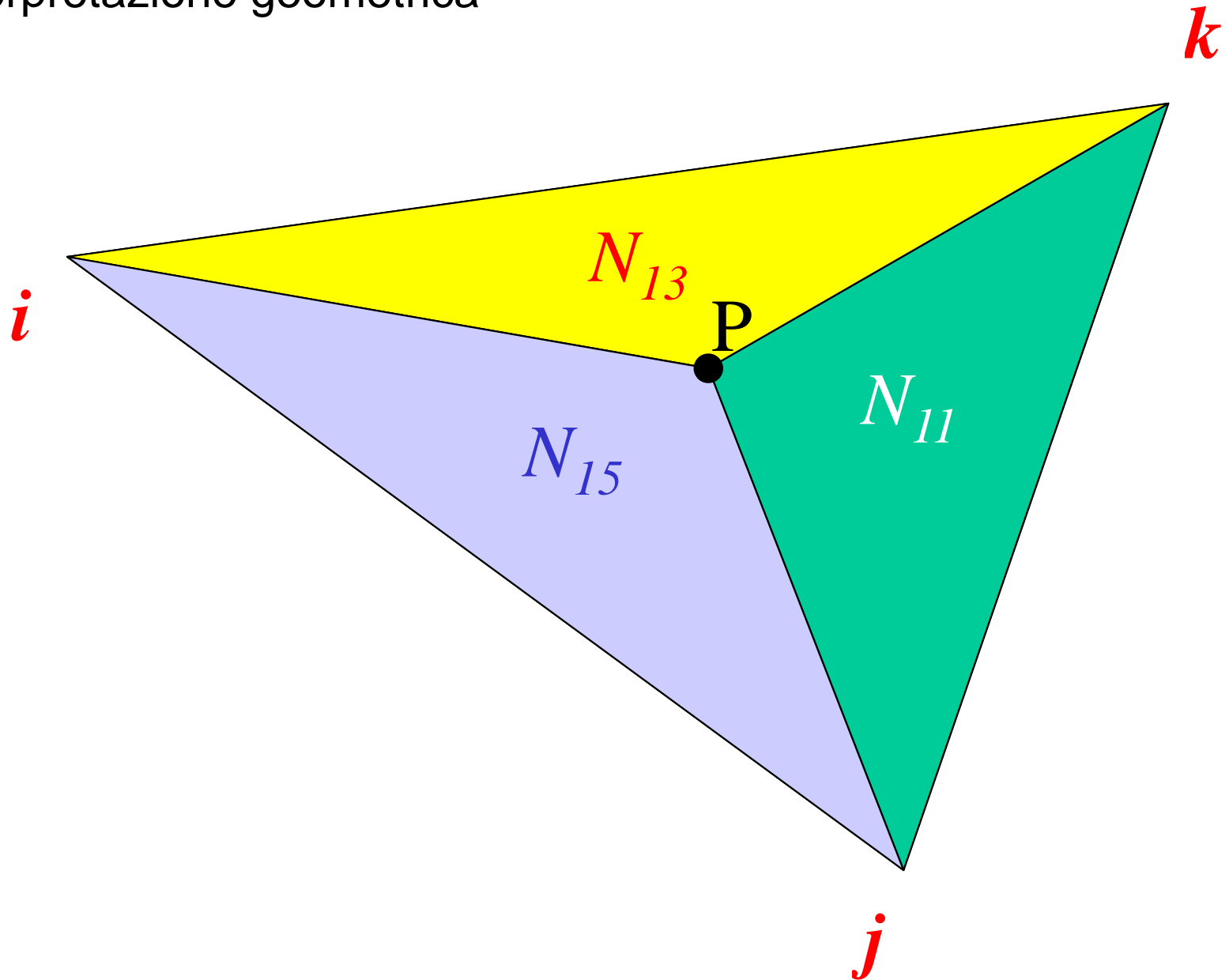
$$\begin{cases} A_{11} = \frac{x_j y_k - x_k y_j}{2\Delta} \\ B_{11} = \frac{y_j - y_k}{2\Delta} \\ C_{11} = \frac{x_k - x_j}{2\Delta} \end{cases} \quad 2\Delta = \det \begin{bmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{bmatrix}$$







# Interpretazione geometrica

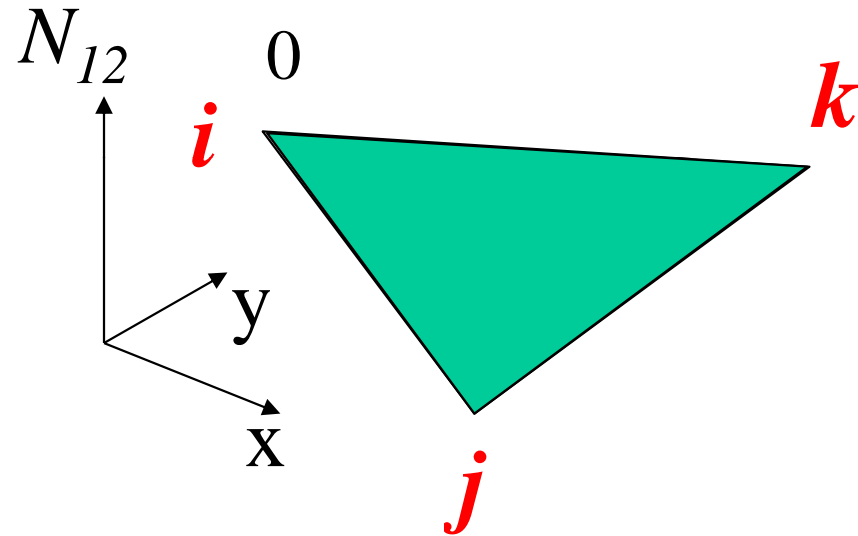


$$\begin{cases} N_{12}(x_i, y_i) = 0 \\ N_{12}(x_j, y_j) = 0 \\ N_{12}(x_k, y_k) = 0 \end{cases}$$

$$N_{lm}^e(x, y) = A_{lm} + B_{lm} \cdot x + C_{lm} \cdot y$$

$$\begin{cases} A_{12} + B_{12}x_i + C_{12}y_i = 0 \\ A_{12} + B_{12}x_j + C_{12}y_j = 0 \\ A_{12} + B_{12}x_k + C_{12}y_k = 0 \end{cases}$$

$$\begin{cases} A_{12} = 0 \\ B_{12} = 0 \\ C_{12} = 0 \end{cases}$$





# Matrice delle funzioni di forma

$$\{v(x, y)\} = \begin{Bmatrix} v_x(x, y) \\ v_y(x, y) \end{Bmatrix} = [N^e(x, y)] \cdot \{U^e\}$$

$2 \times 1$                    $2 \times 1$                    $2 \times 6$                    $6 \times 1$



$$\begin{bmatrix} N_{11}(x, y) & 0 & N_{13}(x, y) & 0 & N_{15}(x, y) & 0 \\ 0 & N_{22} = N_{11} & 0 & N_{24} = N_{13} & 0 & N_{26} = N_{15} \end{bmatrix}$$

## Calcolo delle deformazioni

Spostamenti

*congruenza*

Deformazioni

$$\left\{ \begin{array}{l} \varepsilon_x = \frac{\partial v_x}{\partial x} \\ \varepsilon_y = \frac{\partial v_y}{\partial y} \\ \gamma_{xy} = \frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \end{array} \right. \longrightarrow \left\{ \begin{array}{l} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{array} \right\} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \left\{ \begin{array}{l} v_x(x, y) \\ v_y(x, y) \end{array} \right\} = [L]\{v(x, y)\}$$



$$\{\varepsilon(x, y)\} = [L]\{v(x, y)\}$$

3x1

3x2

2x1

$$\{v(x, y)\} = [N(x, y)]\{U^e\}$$

2x1

2x6

6x1

$$\{\varepsilon\} = [L][N]\{U^e\} = [B]\{U^e\}$$

3x1

3x6 6x1

## Contenuto matrice [B]

$$[B] = [L][N] = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} N_{11} & 0 & N_{13} & 0 & N_{15} & 0 \\ 0 & N_{22} & 0 & N_{24} & 0 & N_{26} \end{bmatrix}$$

$$[B] = \begin{bmatrix} \frac{\partial N_{11}}{\partial x} & 0 & \frac{\partial N_{13}}{\partial x} & 0 & \frac{\partial N_{15}}{\partial x} & 0 \\ 0 & \frac{\partial N_{22}}{\partial y} & 0 & \frac{\partial N_{24}}{\partial y} & 0 & \frac{\partial N_{26}}{\partial y} \\ \frac{\partial N_{11}}{\partial y} & \frac{\partial N_{22}}{\partial x} & \frac{\partial N_{13}}{\partial y} & \frac{\partial N_{24}}{\partial x} & \frac{\partial N_{15}}{\partial y} & \frac{\partial N_{26}}{\partial x} \end{bmatrix}$$





$$N_{11} = A_{11} + B_{11}x + C_{11}y$$

$$\frac{\partial N_{11}}{\partial x} = B_{11} = \frac{y_j - y_k}{2\Delta}$$

$$\frac{\partial N_{11}}{\partial y} = C_{11} = \frac{x_k - x_j}{2\Delta}$$

$$[B] = \begin{bmatrix} B_{11} & 0 & B_{13} & 0 & B_{15} & 0 \\ 0 & C_{22} & 0 & C_{24} & 0 & C_{26} \\ C_{11} & B_{22} & C_{13} & B_{24} & C_{15} & B_{26} \end{bmatrix}$$

# Relazioni costitutive

Esempio 1: stato piano di tensione, materiale isotropo

$$\left\{ \begin{array}{l} \varepsilon_x = \frac{\sigma_x}{E} - \frac{\nu\sigma_y}{E} \\ \varepsilon_y = \frac{\sigma_y}{E} - \frac{\nu\sigma_x}{E} \\ \gamma_{xy} = \frac{2(1+\nu)\tau_{xy}}{E} \end{array} \right. \quad \rightarrow \quad \left\{ \begin{array}{l} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{array} \right\} = \frac{E}{1-\nu^2} \left[ \begin{array}{ccc} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{array} \right] \left\{ \begin{array}{l} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{array} \right\}$$

$$\{\sigma\} = [D]\{\varepsilon\}$$



# Relazioni costitutive

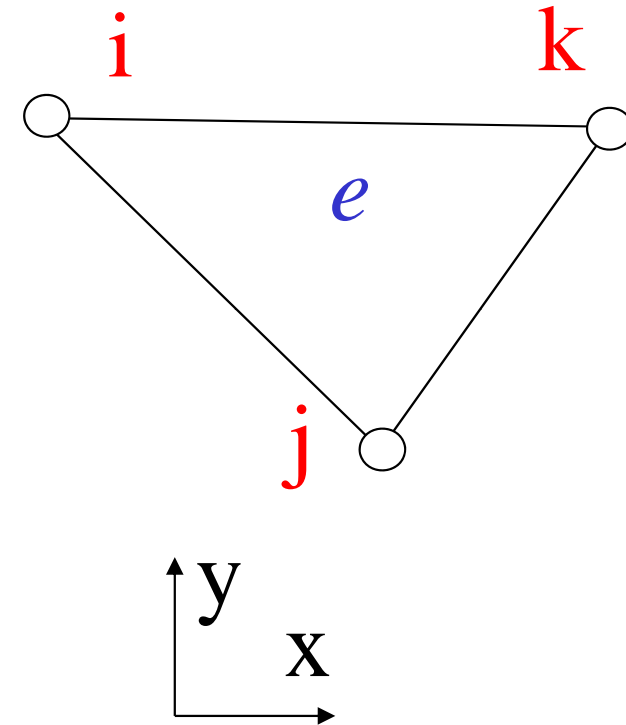
Esempio 2: stato piano di deformazione, materiale isotropo

$$\left\{ \begin{array}{l} \varepsilon_x = \frac{\sigma_x}{E} - \frac{\nu\sigma_y}{E} - \frac{\nu\sigma_z}{E} \\ \varepsilon_y = \frac{\sigma_y}{E} - \frac{\nu\sigma_x}{E} - \frac{\nu\sigma_z}{E} \\ \gamma_{xy} = \frac{2(1+\nu)\tau_{xy}}{E} \end{array} \right. \quad [D] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & (1-2\nu)/2 \end{bmatrix}$$

$$\varepsilon_z = \frac{\sigma_z}{E} - \frac{\nu\sigma_x}{E} - \frac{\nu\sigma_y}{E} = 0$$

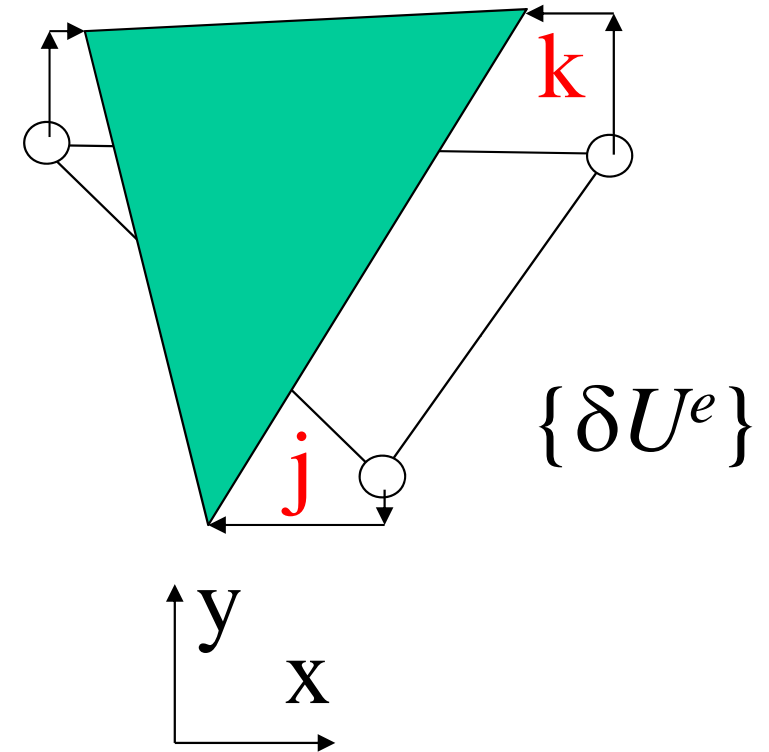
Valutazione di  $[K^e]$ 

## Principio dei Lavori Virtuali



Valutazione di  $[K^e]$ 

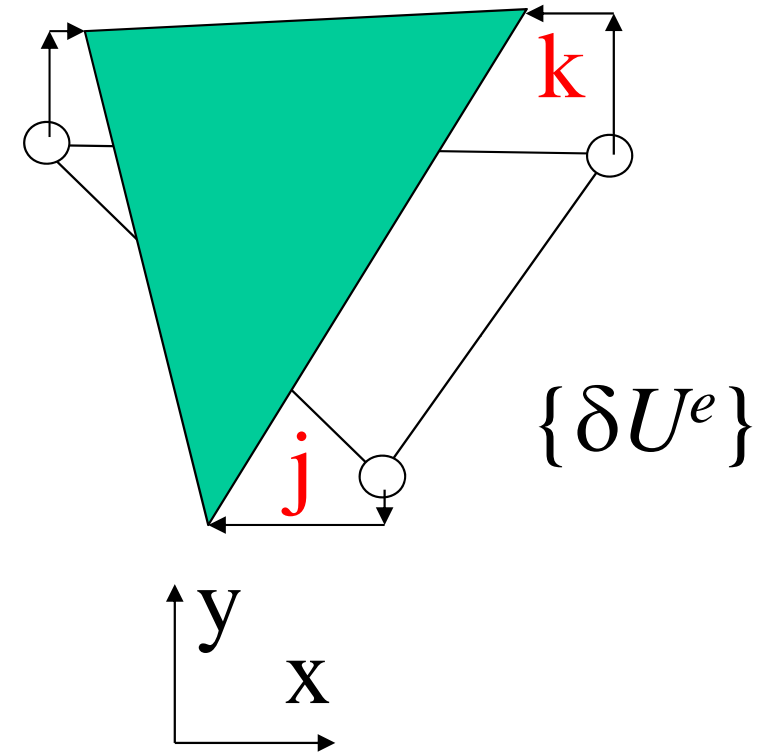
## Principio dei Lavori Virtuali



Valutazione di  $[K^e]$ 

Principio dei Lavori Virtuali

$$L_{est} = L_{int}$$



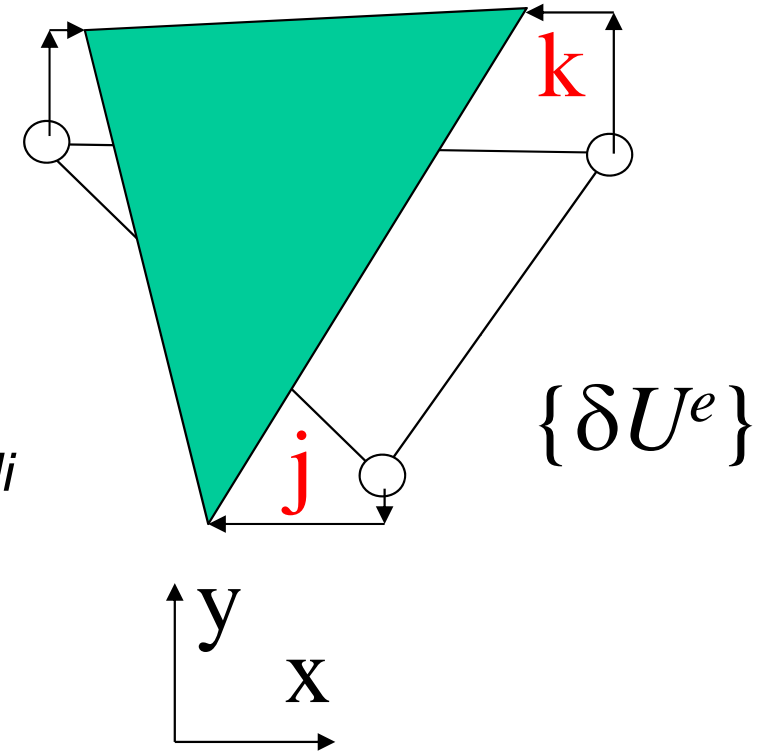
Valutazione di  $[K^e]$ 

## Principio dei Lavori Virtuali

$$L_{est} = L_{int}$$

*Carichi nodali veri \*  
spost.nodali virtuali*

*Tensioni vere \*  
deformazioni virtuali*



Valutazione di  $[K^e]$ 

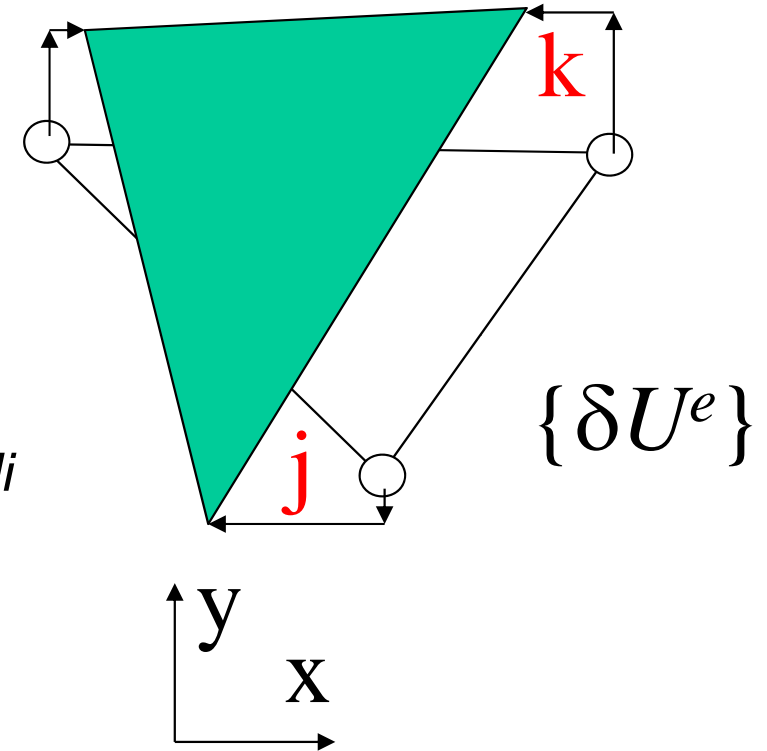
## Principio dei Lavori Virtuali

$$L_{est} = L_{int}$$

*Carichi nodali veri \*  
spost.nodali virtuali*

*Tensioni vere \*  
deformazioni virtuali*

$$L_{est} = \{\delta U^e\}^T \{P^e\}$$





Valutazione di  $[K^e]$ 

## Principio dei Lavori Virtuali

$$L_{est} = L_{int}$$

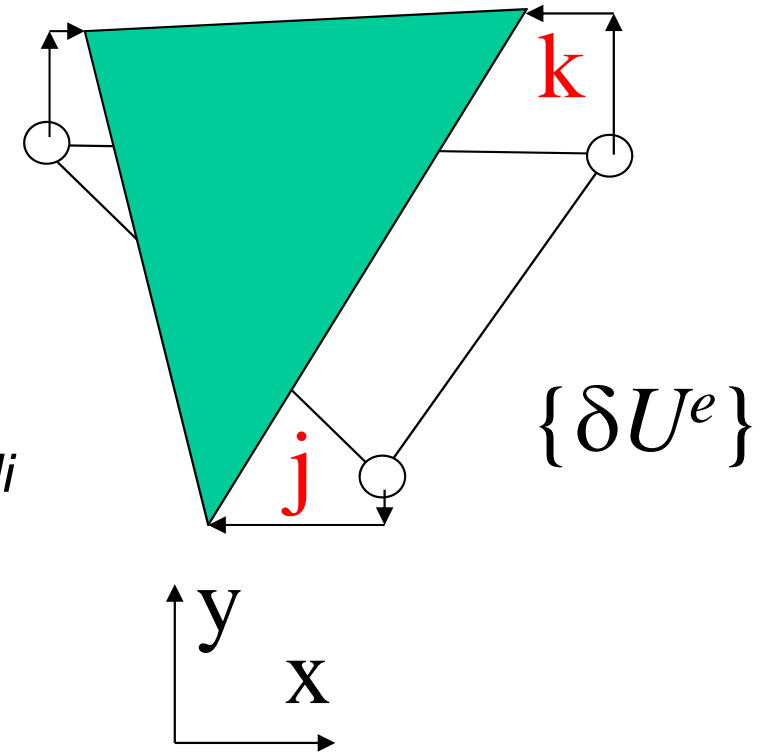
Carichi nodali veri \*  
spost.nodali virtuali

Tensioni vere \*  
deformazioni virtuali

$$L_{est} = \{\delta U^e\}^T \{P^e\}$$

Spost. virtuali

Carichi effettivi





$$L_{\text{int}} = \int_V \{\delta\varepsilon\}^T \{\sigma\} dV$$



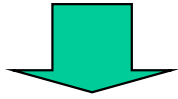
$$L_{\text{int}} = \int_V \{\delta\varepsilon\}^T \{\sigma\} dV$$

$$\{\delta\varepsilon\} = [B]\{\delta U^e\}$$

$$\{\delta\varepsilon\}^T = \{\delta U^e\}^T [B]^T$$



$$L_{\text{int}} = \int_V \{\delta\varepsilon\}^T \{\sigma\} dV$$



$$L_{\text{int}} = \int_V \{\delta U^e\}^T [B]^T \{\sigma\} dV$$

$$\{\delta\varepsilon\} = [B]\{\delta U^e\}$$

$$\{\delta\varepsilon\}^T = \{\delta U^e\}^T [B]^T$$



$$L_{\text{int}} = \int_V \{\delta\varepsilon\}^T \{\sigma\} dV$$

$$\{\delta\varepsilon\} = [B]\{\delta U^e\}$$

$$\{\delta\varepsilon\}^T = \{\delta U^e\}^T [B]^T$$

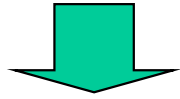
$$L_{\text{int}} = \int_V \{\delta U^e\}^T [B]^T \{\sigma\} dV = \{\delta U^e\}^T \int_V [B]^T \{\sigma\} dV$$



$$L_{\text{int}} = \int_V \{\delta\varepsilon\}^T \{\sigma\} dV$$

$$\{\delta\varepsilon\} = [B]\{\delta U^e\}$$

$$\{\delta\varepsilon\}^T = \{\delta U^e\}^T [B]^T$$



$$L_{\text{int}} = \int_V \{\delta U^e\}^T [B]^T \{\sigma\} dV = \{\delta U^e\}^T \int_V [B]^T \{\sigma\} dV$$

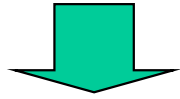
$$\{\sigma\} = [D]\{\varepsilon\}$$



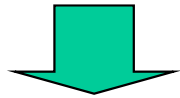
$$L_{\text{int}} = \int_V \{\delta\varepsilon\}^T \{\sigma\} dV$$

$$\{\delta\varepsilon\} = [B]\{\delta U^e\}$$

$$\{\delta\varepsilon\}^T = \{\delta U^e\}^T [B]^T$$



$$L_{\text{int}} = \int_V \{\delta U^e\}^T [B]^T \{\sigma\} dV = \{\delta U^e\}^T \int_V [B]^T \{\sigma\} dV$$



$$L_{\text{int}} = \{\delta U^e\}^T \int_V [B]^T [D] \{\varepsilon\} dV$$

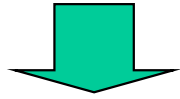
$$\{\sigma\} = [D]\{\varepsilon\}$$



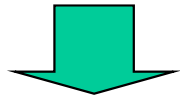
$$L_{\text{int}} = \int_V \{\delta\varepsilon\}^T \{\sigma\} dV$$

$$\{\delta\varepsilon\} = [B]\{\delta U^e\}$$

$$\{\delta\varepsilon\}^T = \{\delta U^e\}^T [B]^T$$

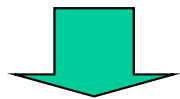


$$L_{\text{int}} = \int_V \{\delta U^e\}^T [B]^T \{\sigma\} dV = \{\delta U^e\}^T \int_V [B]^T \{\sigma\} dV$$



$$L_{\text{int}} = \{\delta U^e\}^T \int_V [B]^T [D] \{\varepsilon\} dV$$

$$\{\sigma\} = [D]\{\varepsilon\}$$



$$L_{\text{int}} = \{\delta U^e\}^T \int_V [B]^T [D] [B] \{U^e\} dV$$

$$\{\varepsilon\} = [B]\{U^e\}$$

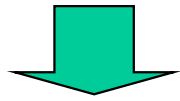




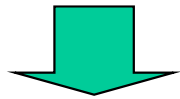
$$L_{\text{int}} = \int_V \{\delta\varepsilon\}^T \{\sigma\} dV$$

$$\{\delta\varepsilon\} = [B]\{\delta U^e\}$$

$$\{\delta\varepsilon\}^T = \{\delta U^e\}^T [B]^T$$



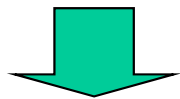
$$L_{\text{int}} = \int_V \{\delta U^e\}^T [B]^T \{\sigma\} dV = \{\delta U^e\}^T \int_V [B]^T \{\sigma\} dV$$



$$L_{\text{int}} = \{\delta U^e\}^T \int_V [B]^T [D] \{\varepsilon\} dV$$

$$\{\sigma\} = [D]\{\varepsilon\}$$

$$\{\varepsilon\} = [B]\{U^e\}$$



$$L_{\text{int}} = \{\delta U^e\}^T \int_V [B]^T [D] [B] \{U^e\} dV = \{\delta U^e\}^T \int_V [B]^T [D] [B] dV \{U^e\}$$



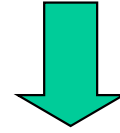
$$L_{est} = \left\{ \delta U^e \right\}^T \left\{ P^e \right\}$$

$$L_{int} = \left\{ \delta U^e \right\}^T \int_V [B]^T [D] [B] dV \left\{ U^e \right\}$$



$$L_{est} = \{\delta U^e\}^T \{P^e\}$$

$$L_{int} = \{\delta U^e\}^T \int_V [B]^T [D] [B] dV \{U^e\}$$




$$\{\delta U^e\}^T \{P^e\} = \{\delta U^e\}^T \int_V [B]^T [D] [B] dV \{U^e\}$$



$$L_{est} = \{\delta U^e\}^T \{P^e\}$$

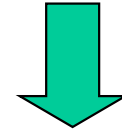
$$L_{int} = \{\delta U^e\}^T \int_V [B]^T [D] [B] dV \{U^e\}$$


$$\{\delta U^e\}^T \{P^e\} = \{\delta U^e\}^T \int_V [B]^T [D] [B] dV \{U^e\}$$



$$L_{est} = \{\delta U^e\}^T \{P^e\}$$

$$L_{int} = \{\delta U^e\}^T \int_V [B]^T [D] [B] dV \{U^e\}$$



$$\{\delta U^e\}^T \{P^e\} = \{\delta U^e\}^T \int_V [B]^T [D] [B] dV \{U^e\}$$

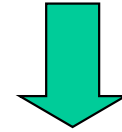


$$\{P^e\} = \int_V [B]^T [D] [B] dV \{U^e\}$$



$$L_{est} = \{\delta U^e\}^T \{P^e\}$$

$$L_{int} = \{\delta U^e\}^T \int_V [B]^T [D] [B] dV \{U^e\}$$



$$\{\delta U^e\}^T \{P^e\} = \{\delta U^e\}^T \int_V [B]^T [D] [B] dV \{U^e\}$$



$$\{P^e\} = \int_V [B]^T [D] [B] dV \{U^e\}$$

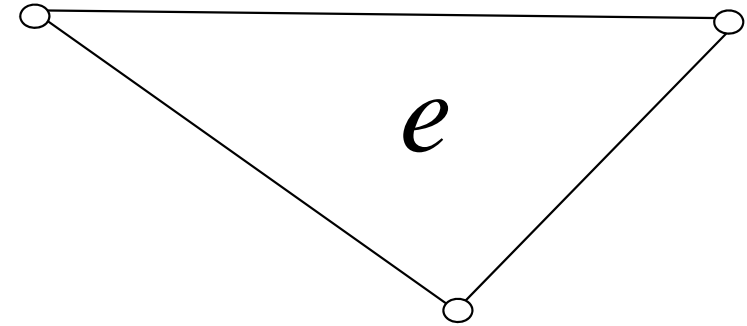


$$\{P^e\} = [K^e] \{U^e\}$$



## Applicazione

$$[K^e] = \int_V [B]^T [D] [B] dV$$

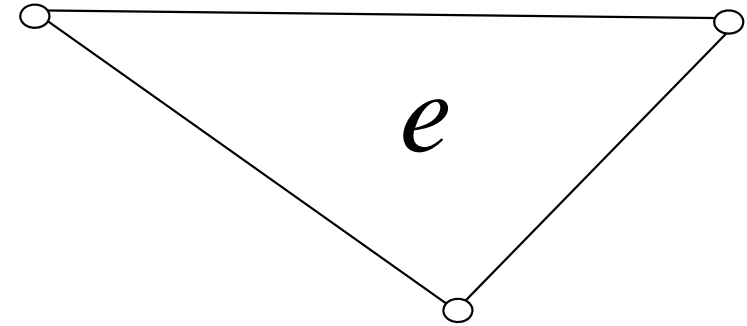




## Applicazione

$$[K^e] = \int_V [B]^T [D] [B] dV$$

$$[B] = \begin{bmatrix} B_{11} & 0 & B_{13} & 0 & B_{15} & 0 \\ 0 & C_{22} & 0 & C_{24} & 0 & C_{26} \\ C_{11} & B_{22} & C_{13} & B_{24} & C_{15} & B_{26} \end{bmatrix}$$





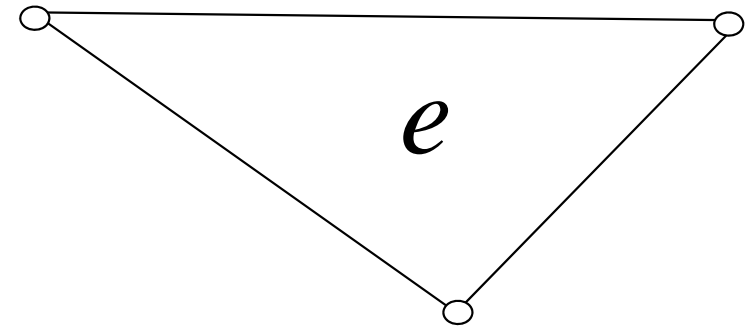


## Applicazione

$$[K^e] = \int_V [B]^T [D] [B] dV$$

$$[B] = \begin{bmatrix} B_{11} & 0 & B_{13} & 0 & B_{15} & 0 \\ 0 & C_{22} & 0 & C_{24} & 0 & C_{26} \\ C_{11} & B_{22} & C_{13} & B_{24} & C_{15} & B_{26} \end{bmatrix}$$

$$[D] = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}$$





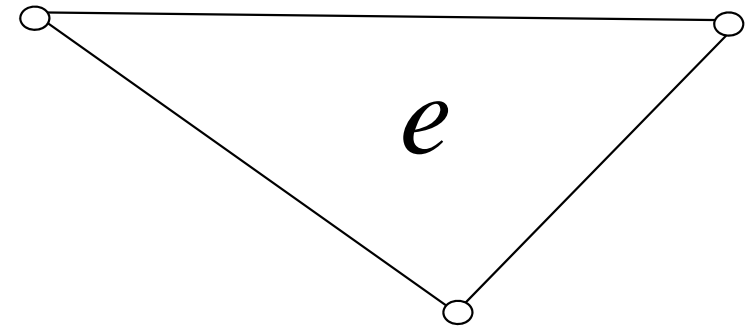
## Applicazione

$$[K^e] = \int_V [B]^T [D] [B] dV$$

$$[B] = \begin{bmatrix} B_{11} & 0 & B_{13} & 0 & B_{15} & 0 \\ 0 & C_{22} & 0 & C_{24} & 0 & C_{26} \\ C_{11} & B_{22} & C_{13} & B_{24} & C_{15} & B_{26} \end{bmatrix}$$

$$[D] = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}$$

$$[K^e] = [B]^T [D] [B] \int_V dV = [B]^T [D] [B] V$$





## Osservazione: unità di misura

$$[K^e] = [B]^T [D] [B] V$$



## Osservazione: unità di misura

$$\text{N m}^{-1} \rightarrow [K^e] = [B]^T [D] [B] V$$



## Osservazione: unità di misura

$$\begin{array}{c} \text{N m}^{-1} \nearrow \\ [K^e] = [B]^T [D] [B] V \\ \nwarrow \nearrow \\ \text{m}^{-1} \quad \text{m}^{-1} \end{array}$$



## Osservazione: unità di misura

$$\begin{array}{c} \text{N m}^{-1} \nearrow \\ [K^e] = [B]^T [D] [B] V \\ \begin{array}{ccc} \nearrow & \uparrow & \nwarrow \\ \text{m}^{-1} & \text{N m}^{-2} & \text{m}^{-1} \end{array} \end{array}$$



## Osservazione: unità di misura

$$[K^e] = [B]^T [D] [B] V$$

Units of measurement are indicated by arrows:

- $[K^e]$  is  $\text{N m}^{-1}$  (blue text)
- $[B]^T$  is  $\text{m}^{-1}$  (red text)
- $[D]$  is  $\text{N m}^{-2}$  (red text)
- $[B]$  is  $\text{m}^{-1}$  (red text)
- $V$  is  $\text{m}^3$  (red text)



## Osservazione: unità di misura

$$\begin{array}{c}
 \text{N m}^{-1} \rightarrow [K^e] = [B]^T [D] [B] V \leftarrow \text{m}^3 \\
 \begin{array}{ccc}
 \nearrow & \uparrow & \nwarrow \\
 \text{m}^{-1} & \text{N m}^{-2} & \text{m}^{-1}
 \end{array}
 \end{array}$$

$$\frac{1}{m} \frac{N}{m^2} \frac{1}{m} m^3 = \frac{N}{m}$$





# Calcolo della matrice $[K^e]$

$$[K^e] = \int_V [B]^T [D] [B] dV$$



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$$[K^e] = \int_V [B]^T [D] [B] dV$$

Integrale calcolato numericamente (Metodo di Gauss)

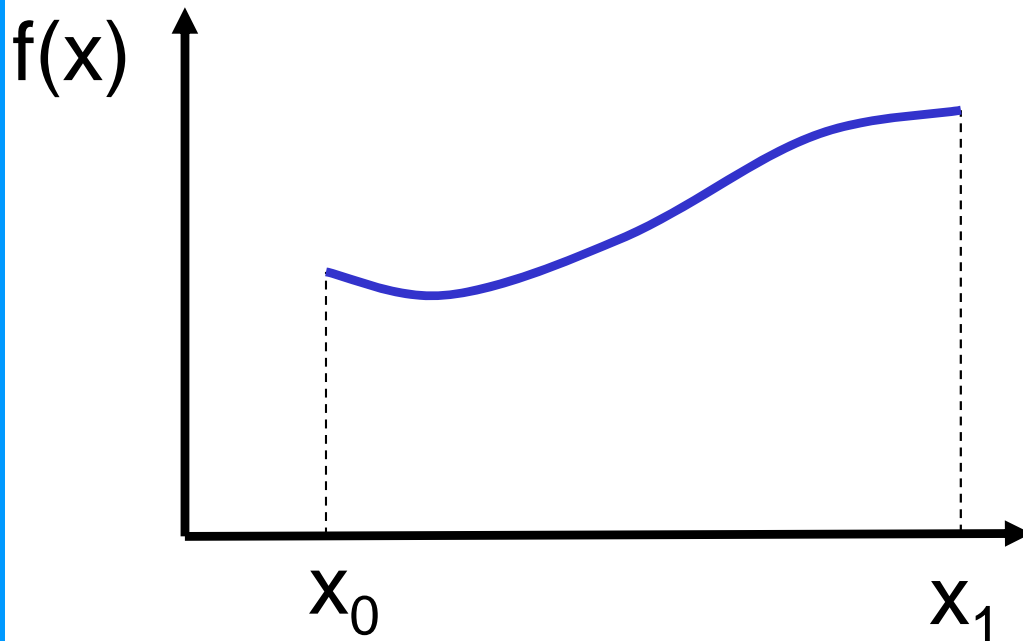


# Calcolo della matrice $[K^e]$

$$[K^e] = \int_V [B]^T [D] [B] dV$$

Integrale calcolato numericamente (Metodo di Gauss)

Metodi classici di integrazione:





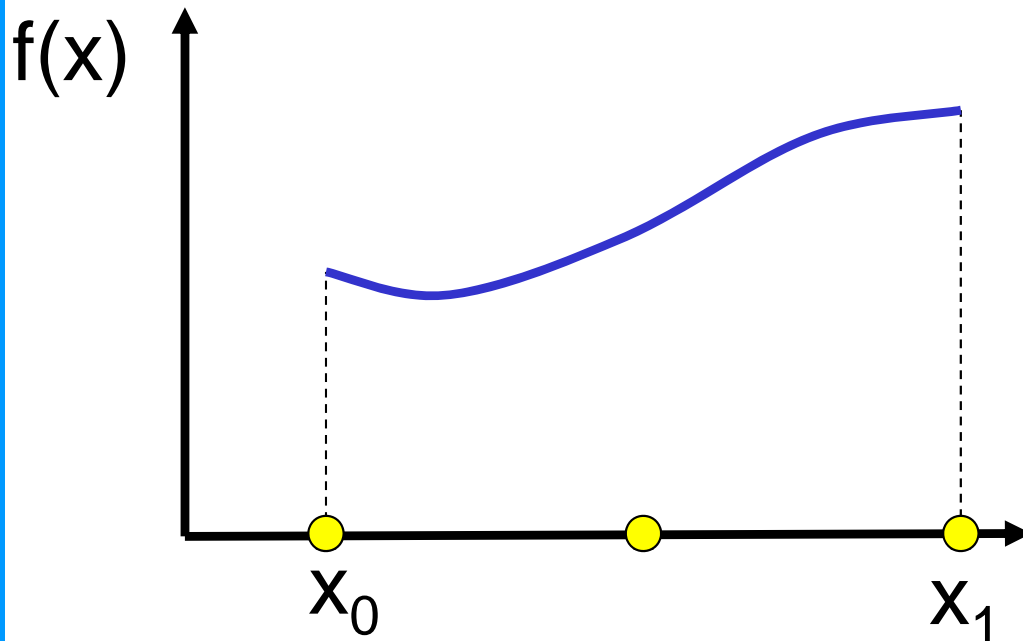
# Calcolo della matrice $[K^e]$

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Metodi classici di integrazione:

1) Si scelgono “a priori”  $n$  punti,  $x_i$





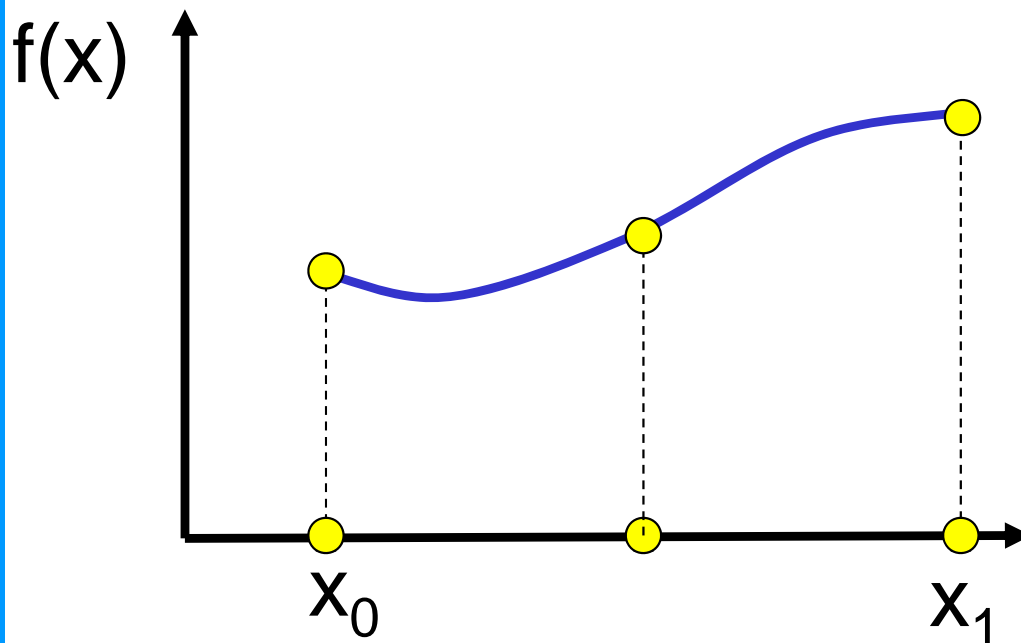
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Metodi classici di integrazione:

- 1) Si scelgono “a priori”  $n$  punti,  $x_i$
- 2) Si calcolano i valori di  $f(x_i)$

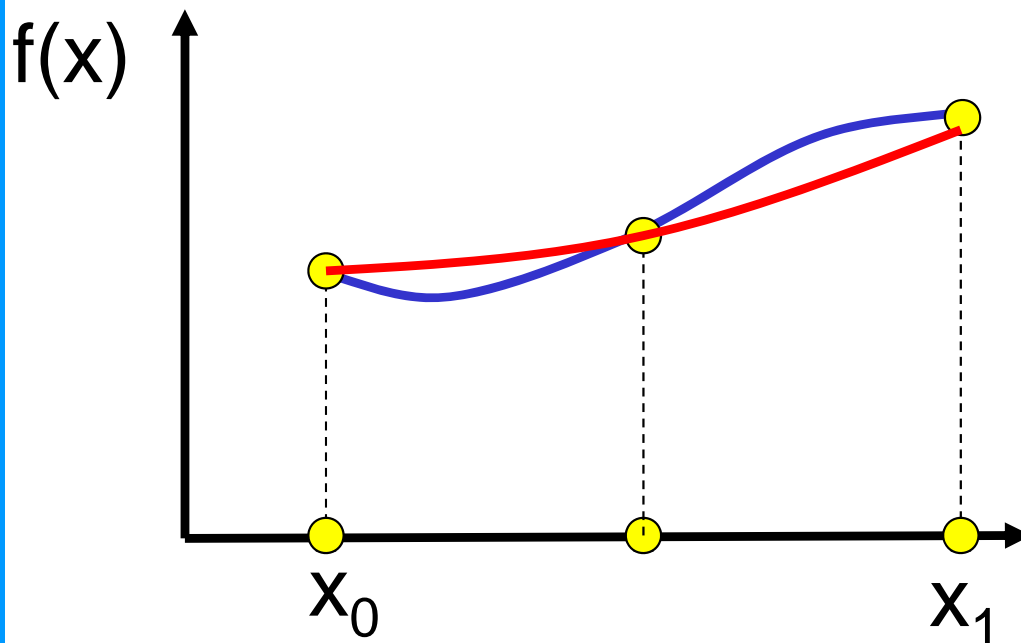


# Calcolo della matrice $[K^e]$

$$[K^e] = \int_V [B]^T [D] [B] dV$$

Integrale calcolato numericamente (Metodo di Gauss)

Metodi classici di integrazione:



- 1) Si scelgono “a priori” n punti,  $x_i$
- 2) Si calcolano i valori di  $f(x_i)$
- 3) Si approssima  $f(x)$  con il polinomio di grado  $n-1$  passante per i punti scelti

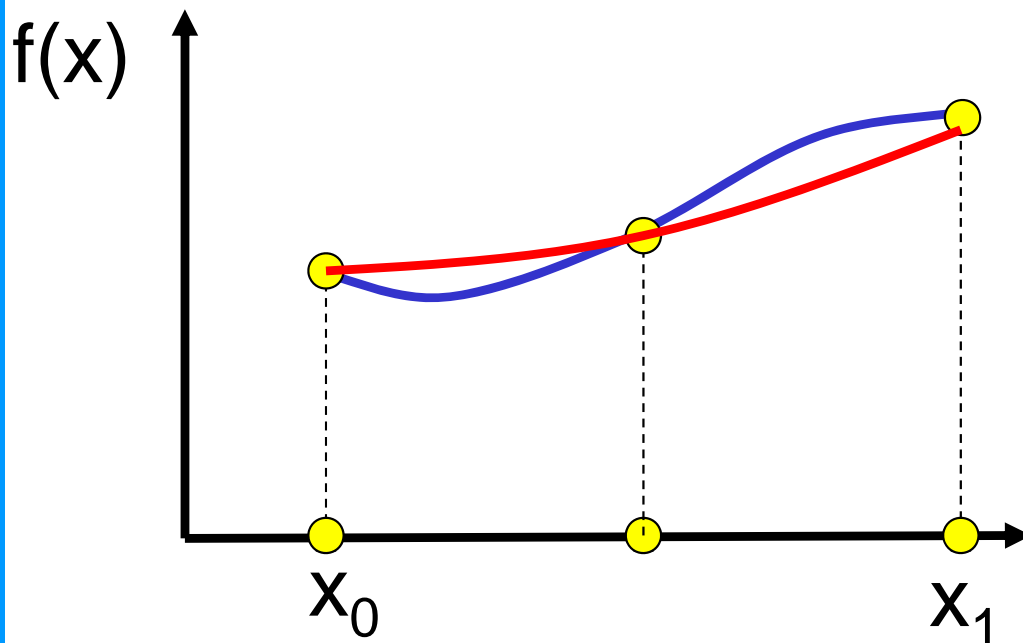


# Calcolo della matrice $[K^e]$

$$[K^e] = \int_V [B]^T [D] [B] dV$$

Integrale calcolato numericamente (Metodo di Gauss)

Metodi classici di integrazione:



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- 4) Si integra il polinomio in forma chiusa

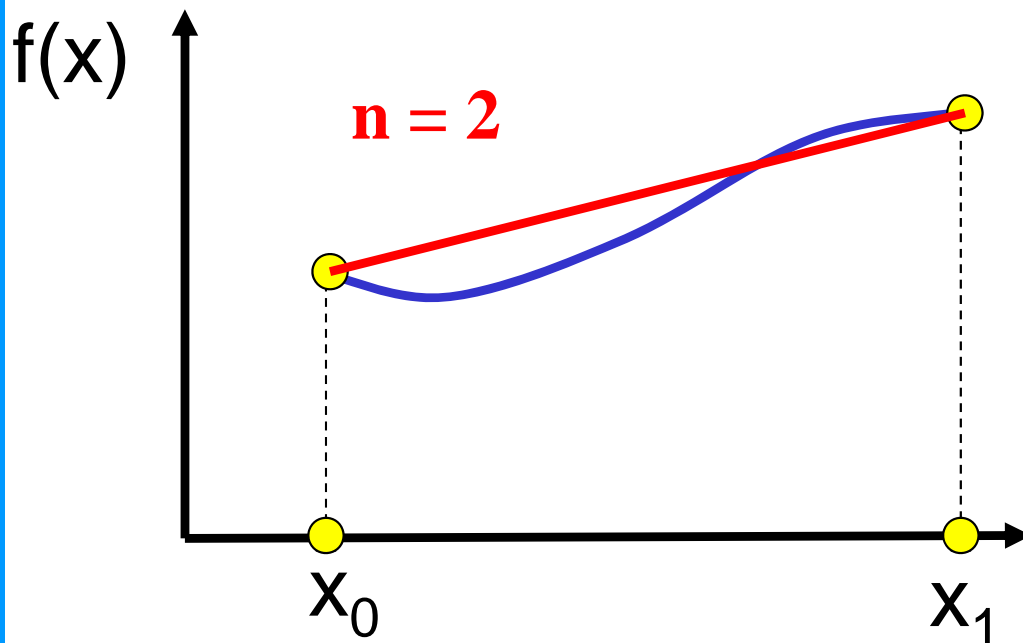
# Calcolo della matrice $[K^e]$

$$[K^e] = \int_V [B]^T [D] [B] dV$$

Integrale calcolato numericamente (Metodo di Gauss)

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- 4) Si integra il polinomio in forma chiusa





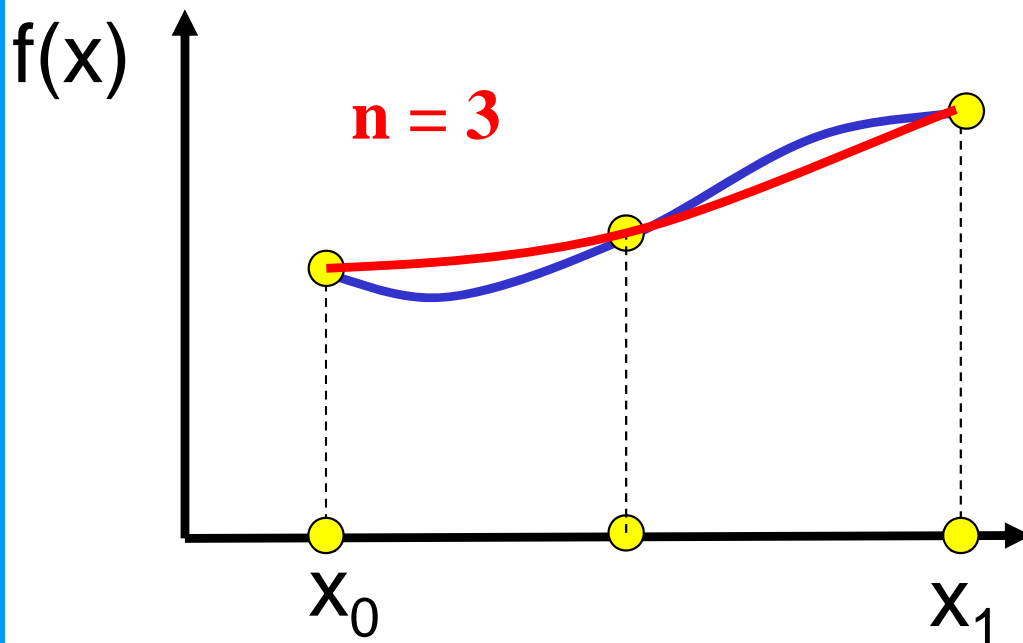
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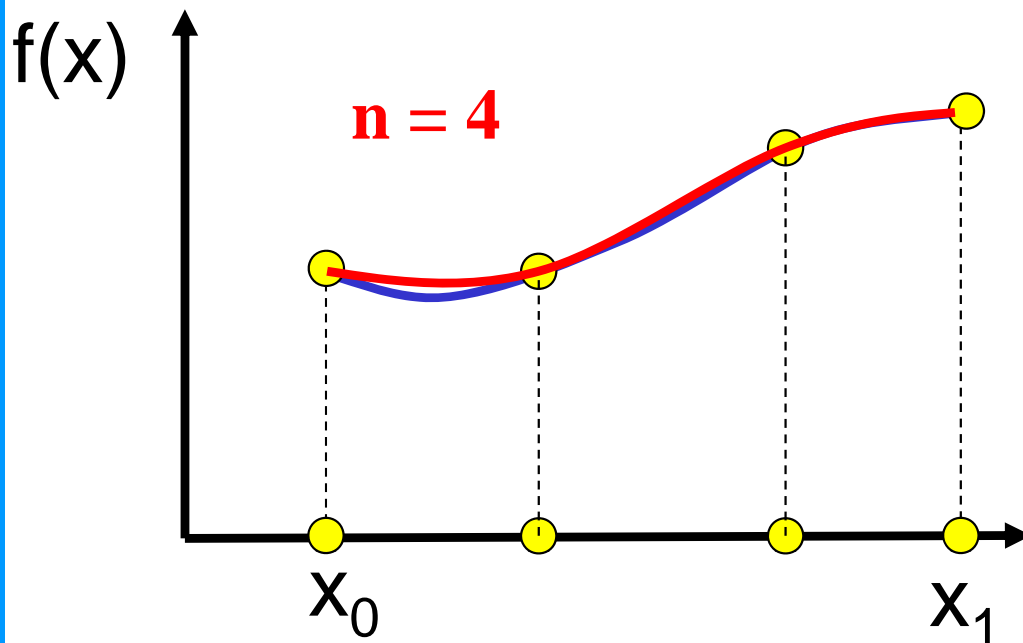


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- 4) Si integra il polinomio in forma chiusa



# Integrazione secondo Gauss: esempio 1D

$$\int_{x_I}^{x_F} f(x) dx \cong \sum_{i=1}^n W_i f(x_i)$$



# Integrazione secondo Gauss: esempio 1D

Integrale da calcolare  $\rightarrow \int_{x_I}^{x_F} f(x) dx \cong \sum_{i=1}^n W_i f(x_i) \leftarrow$  Valore della f.ne nel punto  $x_i$

$\uparrow$   
Peso



# Integrazione secondo Gauss: esempio 1D

Integrale da calcolare  $\rightarrow \int_{x_I}^{x_F} f(x) dx \cong \sum_{i=1}^n W_i f(x_i) \leftarrow$  Valore della f.ne nel punto  $x_i$

↑  
Peso

1) Si fissa n



# Integrazione secondo Gauss: esempio 1D

Integrale da calcolare  $\rightarrow \int_{x_I}^{x_F} f(x) dx \cong \sum_{i=1}^n W_i f(x_i) \leftarrow$  Valore della f.ne nel punto  $x_i$

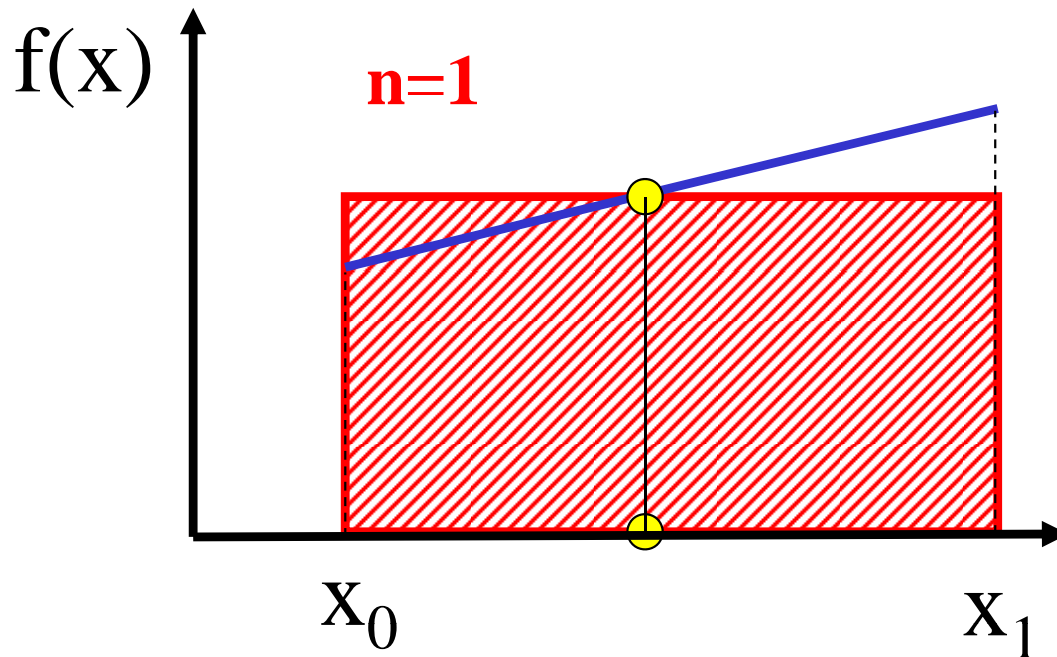
$\uparrow$   
Peso

- 1) Si fissa  $n$
- 2) Si scelgono gli  $x_i$  ed i  $W_i$  in modo da valutare in modo esatto l'integrale di un polinomio di grado  $2n-1$  sull'intervallo dato

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Integrale da calcolare  $\rightarrow \int_{x_I}^{x_F} f(x) dx \cong \sum_{i=1}^n W_i f(x_i) \leftarrow$  Valore della f.ne nel punto  $x_i$

$\uparrow$   
Peso



1) Si fissa  $n$

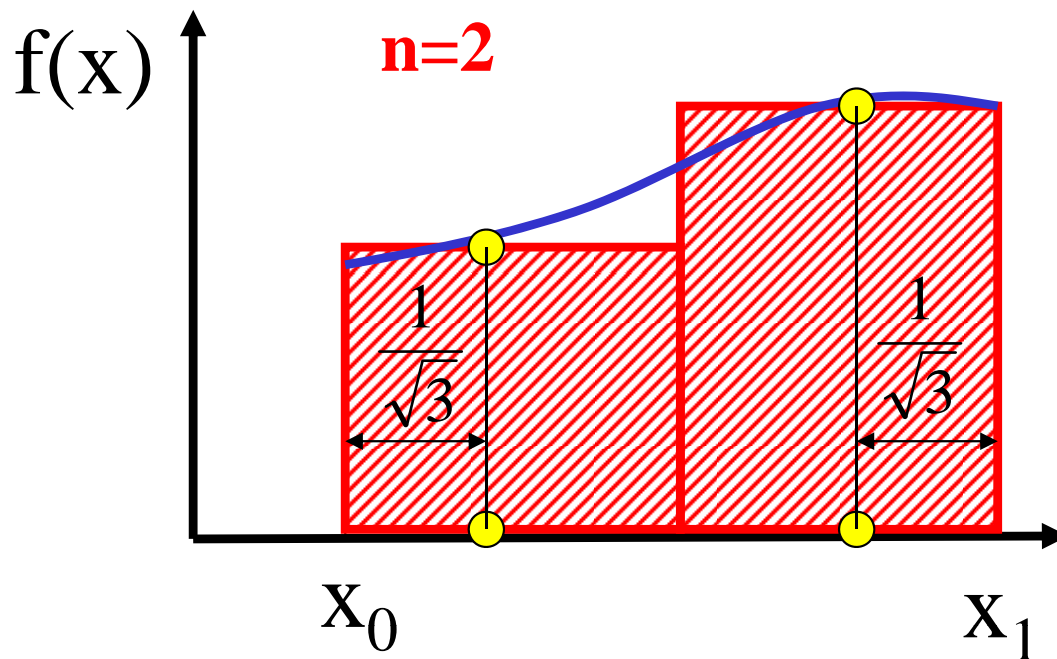
2) Si scelgono gli  $x_i$  ed i  $W_i$  in modo da valutare in modo esatto l'integrale di un polinomio di grado  $2n-1$  sull'intervallo dato

I punti  $x_i$  sono detti "punti di Gauss"

# Integrazione secondo Gauss: esempio 1D

Integrale da calcolare  $\rightarrow \int_{x_I}^{x_F} f(x) dx \cong \sum_{i=1}^n W_i f(x_i) \leftarrow$  Valore della f.ne nel punto  $x_i$

$\uparrow$   
Peso



1) Si fissa  $n$

2) Si scelgono gli  $x_i$  ed i  $W_i$  in modo da valutare in modo esatto l'integrale di un polinomio di grado  $2n-1$  sull'intervallo dato

I punti  $x_i$  sono detti "punti di Gauss"





## Vantaggi dell'integrazione secondo Gauss:

- fissato  $n$ , consente il calcolo esatto dell'integrale di una f.ne di grado  $2n-1$  anziché  $n-1$
- dato il grado  $n$  della f.ne che si vuole poter integrare esattamente, richiede il calcolo della f.ne stessa in  $(n+1)/2$  punti, anziché in  $n+1$  punti

Le posizioni dei punti di Gauss per integrali in 1, 2 e 3 dimensioni sono note per molti domini di integrazione.



# ANALISI INTERA STRUTTURA

Congruenza



[B]

Costitutive



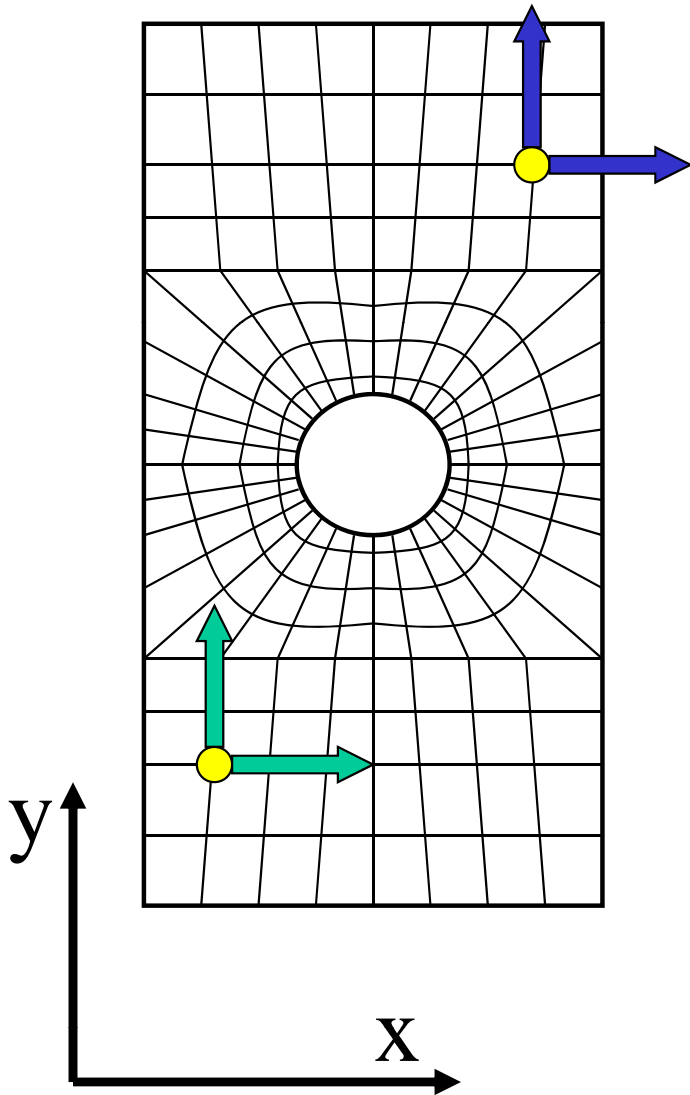
[D]

Equilibrio



Garantito per il singolo elemento (non ancora per la struttura)

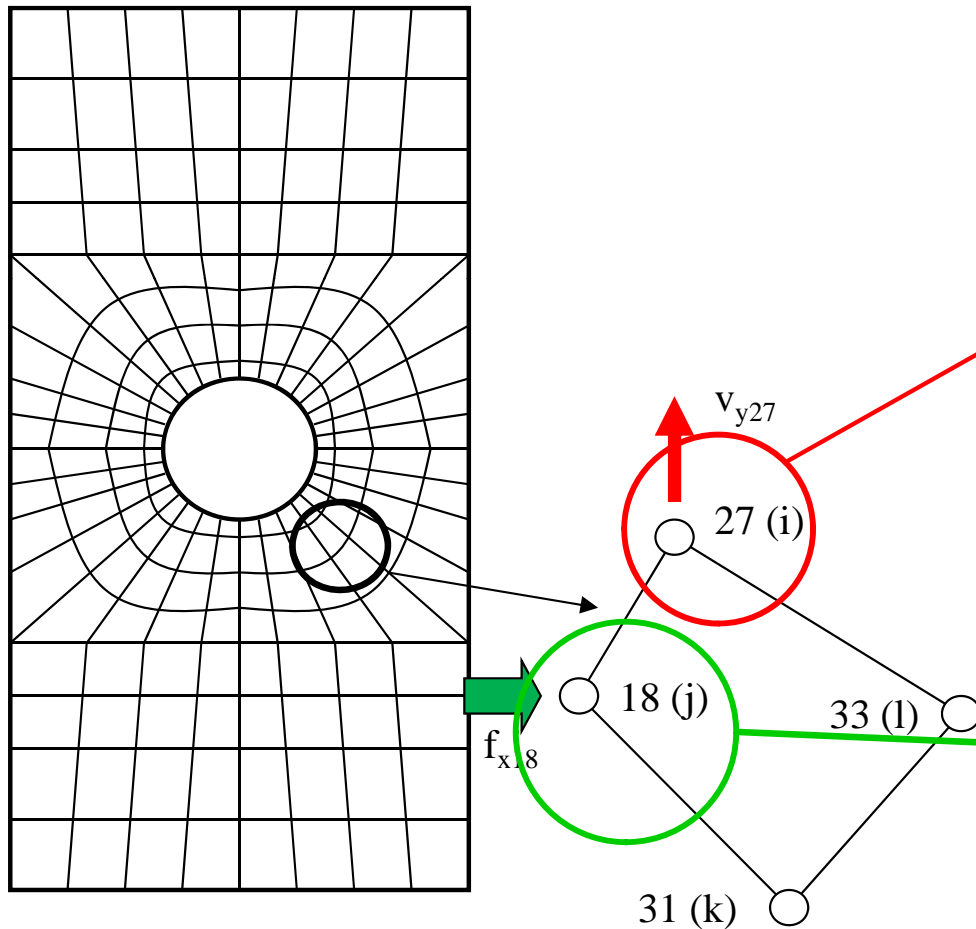
## VETTORI DEGLI SPOSTAMENTI E DEI CARICHI ESTERNI PER L'INTERA STRUTTURA



$$\{U\} = \begin{Bmatrix} v_{x1} \\ v_{y1} \\ v_{x2} \\ - \\ - \\ - \\ v_{yn_N} \end{Bmatrix} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ - \\ - \\ - \\ u_{n_{GDL}} \end{Bmatrix}$$

$$\{F\} = \begin{Bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ - \\ - \\ - \\ f_{yn_N} \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ - \\ - \\ - \\ f_{n_{GDL}} \end{Bmatrix}$$

# VETTORI DEGLI SPOSTAMENTI E DEI CARICHI APPLICATI PER L'INTERA STRUTTURA



$$\{U\} = \begin{Bmatrix} u_1 \\ u_2 \\ - \\ - \\ u_{n_{GDL}} \end{Bmatrix}$$

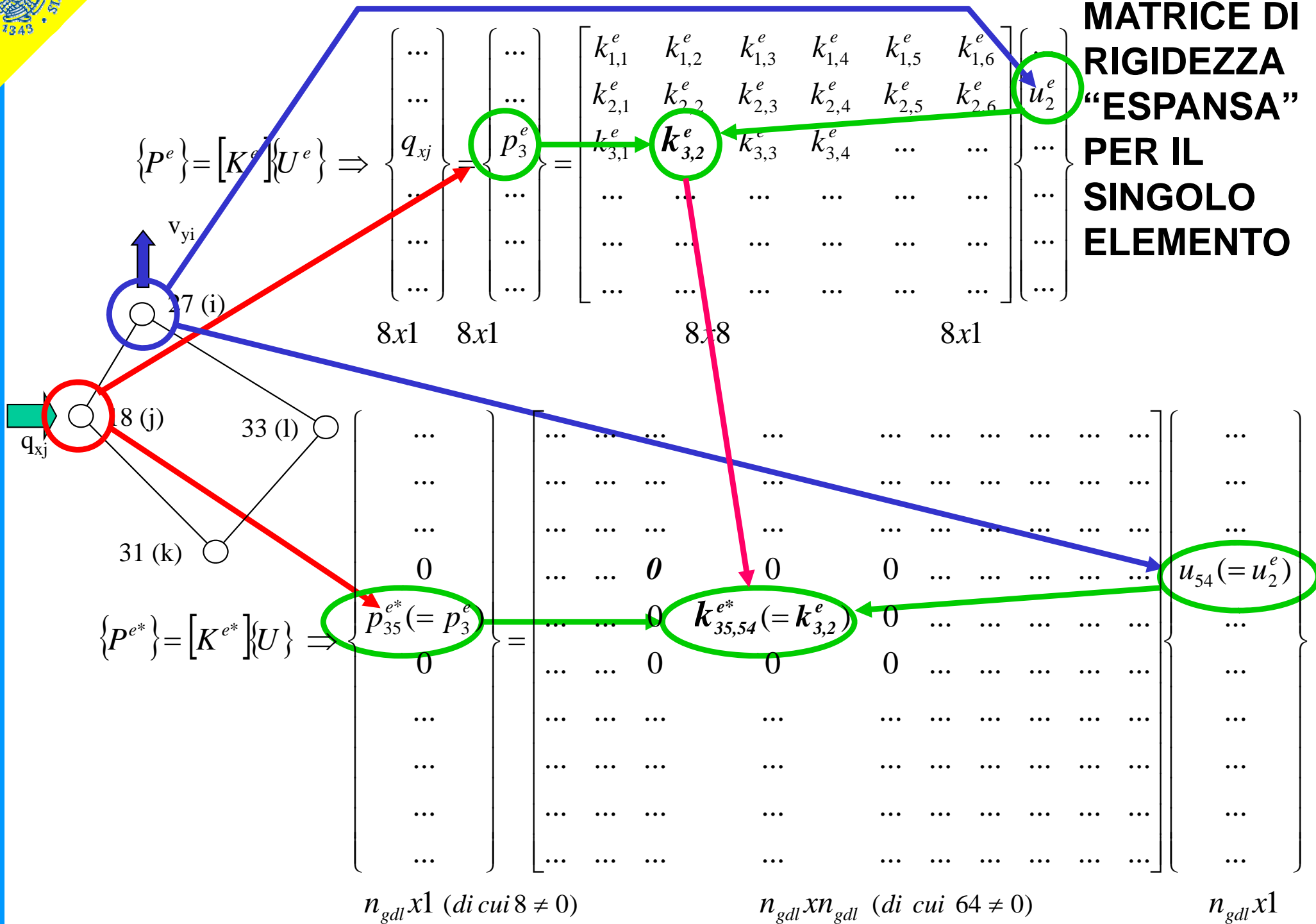
$u_{54} = v_{y27}$

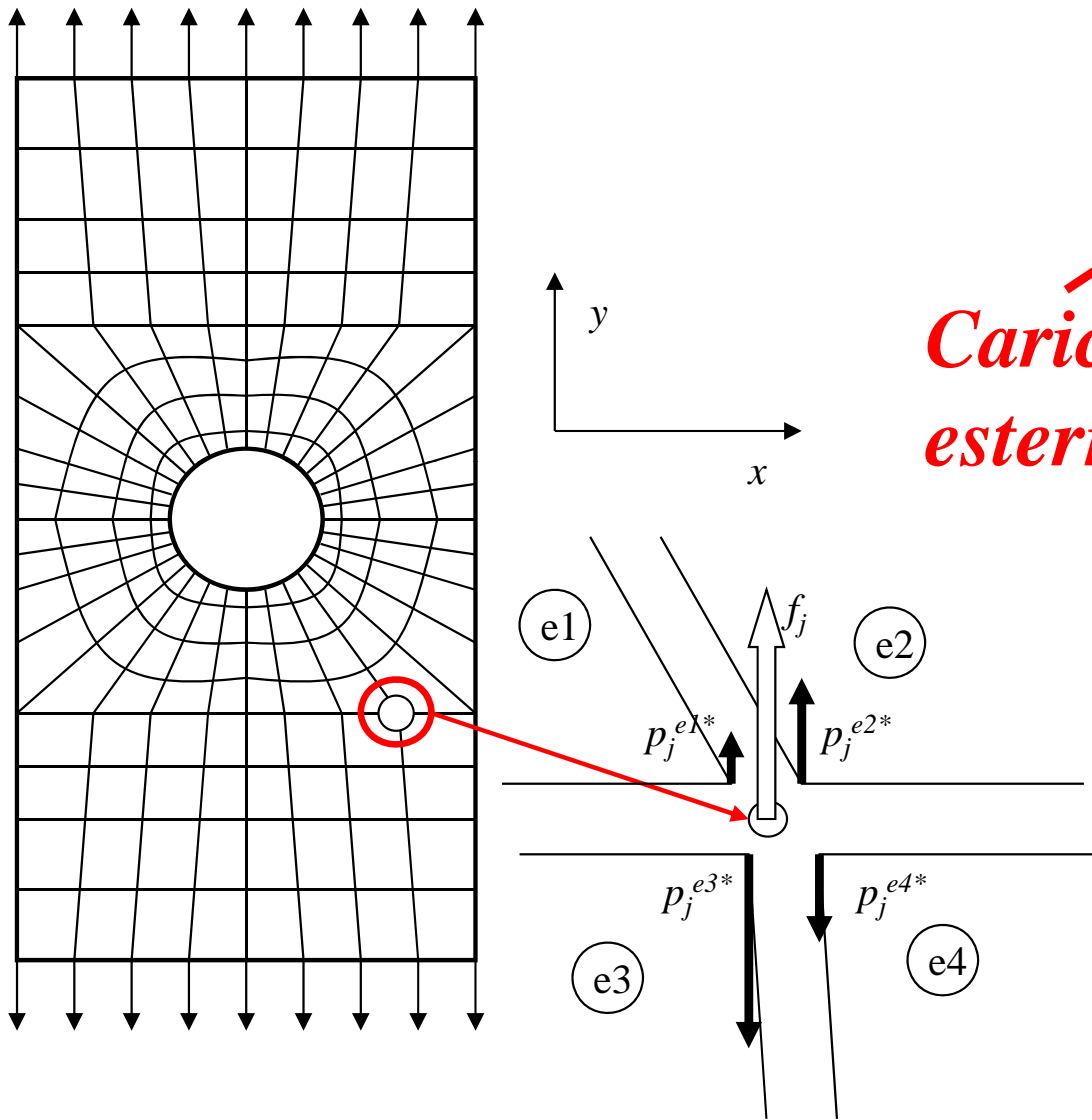
$$\{F\} = \begin{Bmatrix} f_1 \\ f_2 \\ - \\ f_{35} = f_{x18} \\ - \\ - \\ f_{n_{GDL}} \end{Bmatrix}$$

$f_{35} = f_{x18}$



**MATRICE DI RIGIDEZZA "ESPANSA" PER IL SINGOLO ELEMENTO**





*Carico esterno*

$$f_j - \sum_{e=1}^{n_E} p_j^{e*} = 0$$

*Carico applicato nel nodo all'elemento "e"*

$$f_j = \sum_{e=1}^{n_E} p_j^{e*}$$



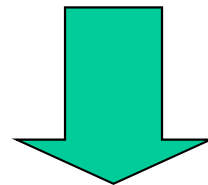
$$\{P^{e*}\} = [K^{e*}]\{U\}$$

$$\begin{aligned}
 f_j &= \sum_{e=1}^{n_E} P_j^{e*} = \sum_{e=1}^{n_E} \left( \sum_{i=1}^{n_{gdl}} k_{ji}^{e*} u_i \right) = \\
 &= \dots + \left( k_{ji}^{1*} + k_{ji}^{2*} + \dots + k_{ji}^{n_E*} \right) u_i + \dots = \\
 &= \sum_{i=1}^{n_{gdl}} \left( \sum_{e=1}^{n_E} k_{ji}^{e*} \right) u_i
 \end{aligned}$$



$$\sum_{e=1}^{n_E} k_{ji}^{e*} = k_{ji}$$

$$f_j = \sum_{i=1}^{n_{gdl}} \left( \sum_{e=1}^{n_E} k_{ji}^{e*} \right) u_i = \sum_{i=1}^{n_{gdl}} k_{ij} u_i$$



*Matrice di rigidezza  
della struttura*

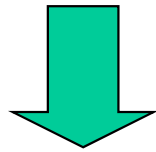
$$\{F\} = [K]\{U\}$$

$n_{GDL} \times 1$        $n_{GDL} \times n_{GDL}$        $n_{GDL} \times 1$



# SOLUZIONE

$$\{F\} = [K]\{U\}$$



$$\{U\} = [K]^{-1}\{F\}$$

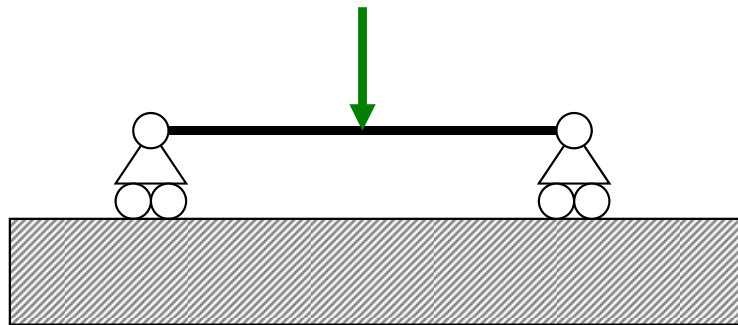
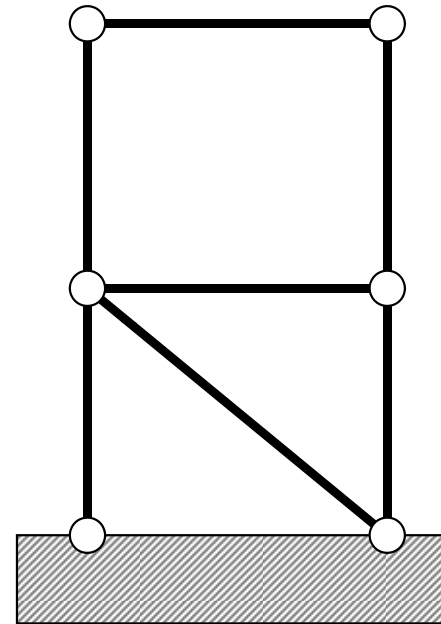
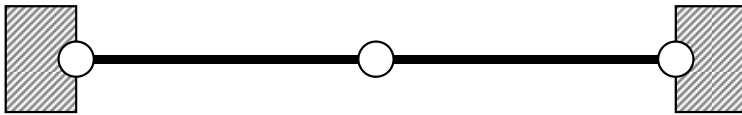
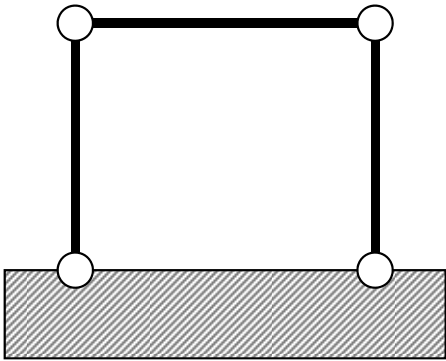
$$\text{c.n.s. : } \det[K] \neq 0$$



$$\det[K] \neq 0$$



Struttura non labile



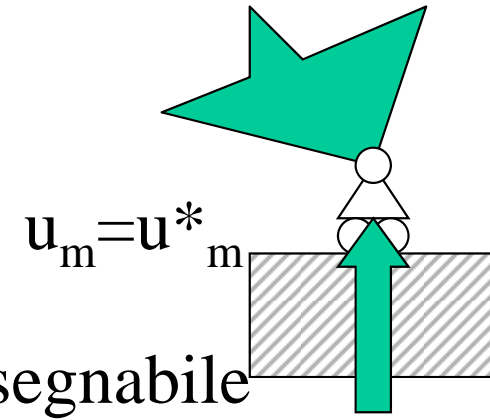
# VINCOLI

Vincolare = assegnare “a priori” il valore di una delle componenti di spostamento (g.d.l.)

$$\begin{array}{c}
 \left\{ \begin{array}{c} f_1 \\ f_2 \\ - \\ - \\ f_m \\ - \\ f_{n_{GDL}} \end{array} \right\} = \begin{bmatrix} k_{11} & k_{12} & - & - & k_{1m} & - & k_{1n_{GDL}} \\ k_{21} & k_{22} & - & - & k_{2m} & - & k_{2n_{GDL}} \\ - & - & - & - & - & - & - \\ - & - & - & - & - & - & - \\ k_{m1} & k_{m2} & - & - & k_{m,m} & - & k_{mn_{GDL}} \\ - & - & - & - & - & - & - \\ k_{n_{GDL}1} & k_{n_{GDL}2} & - & - & k_{n_{GDL}m} & - & k_{n_{GDL}n_{GDL}} \end{bmatrix} \left\{ \begin{array}{c} u_1 \\ u_2 \\ - \\ - \\ u_m \\ - \\ u_{n_{GDL}} \end{array} \right\}
 \end{array}$$

$n_{GDL} \cdot 1$ 
 $n_{GDL} \cdot n_{GDL}$ 
 $n_{GDL} \cdot 1$

$u_m = u^*_m$



$f_m$  non assegnabile

$$\begin{Bmatrix} f_1 \\ f_2 \\ - \\ - \\ f_m \\ - \\ f_{n_{GDL}} \end{Bmatrix} = \begin{Bmatrix} k_{1m} \\ k_{2m} \\ - \\ - \\ k_{m,m} \\ - \\ k_{n_{GDL}m} \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} & - & k_{1m-1} & k_{1m+1} & - & k_{1n_{GDL}} \\ k_{21} & k_{22} & - & k_{2m-1} & k_{2m+1} & - & k_{2n_{GDL}} \\ - & - & - & - & - & - & - \\ - & - & - & - & - & - & - \\ k_{m1} & k_{m2} & - & k_{m,m-1} & k_{m,m+1} & - & k_{mn_{GDL}} \\ - & - & - & - & - & - & - \\ k_{n_{GDL}1} & k_{n_{GDL}2} & - & k_{n_{GDL}m-1} & k_{n_{GDL}m+1} & - & k_{n_{GDL}n_{GDL}} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ - \\ u_{m-1} \\ u_{m+1} \\ - \\ u_{n_{GDL}} \end{Bmatrix}$$

$n_{GDL} \cdot 1$                        $n_{GDL} \cdot (n_{GDL} - 1)$                        $(n_{GDL} - 1) \cdot 1$



# Introduzione vincolo = riduzione di 1 del numero di incognite ed equazioni

$$\begin{array}{c}
 \left. \begin{array}{c} f_1 \\ f_2 \\ - \\ f_{m-1} \\ f_{m+1} \\ - \\ f_{n_{GDL}} \end{array} \right\} \begin{array}{c} - \\ - \\ - \\ - \\ - \\ - \\ - \end{array} u_m \left. \begin{array}{c} k_{1m} \\ k_{2m} \\ - \\ k_{m-1,m} \\ k_{m+1,m} \\ - \\ k_{n_{GDL}m} \end{array} \right\} = \left[ \begin{array}{cccccc} k_{11} & k_{12} & - & k_{1m-1} & k_{1m+1} & - & k_{1n_{GDL}} \\ k_{21} & k_{22} & - & k_{2m-1} & k_{2m+1} & - & k_{2n_{GDL}} \\ - & - & - & - & - & - & - \\ k_{m-1,1} & k_{m-1,1} & - & k_{m-1,m-1} & k_{m-1,m+1} & - & k_{m-1,n_{GDL}} \\ k_{m+1,1} & k_{m+1,2} & - & k_{m+1,m-1} & k_{m+1,m+1} & - & k_{m+1,n_{GDL}} \\ - & - & - & - & - & - & - \\ k_{n_{GDL}1} & k_{n_{GDL}2} & - & k_{n_{GDL}m-1} & k_{n_{GDL}m+1} & - & k_{n_{GDL}n_{GDL}} \end{array} \right] \left. \begin{array}{c} u_1 \\ u_2 \\ - \\ u_{m-1} \\ u_{m+1} \\ - \\ u_{n_{GDL}} \end{array} \right\}
 \end{array}$$

$(n_{GDL}-1) \cdot 1$ 
 $(n_{GDL}-1) \cdot (n_{GDL}-1)$ 
 $(n_{GDL}-1) \cdot 1$



$$[K] = \begin{bmatrix} X & X & X & X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & X & X & X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & X & X & X & X & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & X & 0 & X & X & 0 & 0 & 0 & 0 & 0 \\ & & & & X & X & X & X & 0 & 0 & 0 & 0 \\ & & & & & X & X & 0 & X & 0 & 0 & 0 \\ & & & & & & X & X & X & 0 & 0 & 0 \\ & & & & & & & X & X & X & X & 0 \\ S & I & M & M & . & & & X & X & X & X \\ & & & & & & & & X & X & X \\ & & & & & & & & & X & X \\ & & & & & & & & & & X \end{bmatrix}$$

La matrice  $[K]$ :

- è simmetrica
- ha una struttura “a banda” attorno alla diagonale principale



Esistono molti metodi di soluzione del sistema. Uno dei più comuni ed efficienti è il metodo di eliminazione diretta di Gauss.

$$\begin{bmatrix}
 X & X & X & X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 X & X & X & X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 X & X & X & X & X & X & 0 & 0 & 0 & 0 & 0 & 0 \\
 X & X & X & X & 0 & X & X & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & X & 0 & X & X & X & X & 0 & 0 & 0 & 0 \\
 0 & 0 & X & X & X & X & X & 0 & X & 0 & 0 & 0 \\
 0 & 0 & 0 & X & X & X & X & X & X & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & X & 0 & X & X & X & X & X & 0 \\
 0 & 0 & 0 & 0 & 0 & X & X & X & X & X & X & X \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & X & X & X & X & X \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & X & X & X & X & X \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & X & X & X & X \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & X & X \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & X
 \end{bmatrix}
 \quad
 \begin{bmatrix}
 X & X & X & X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & X & X & X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & -0 & -X & X & X & X & X & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & X & 0 & X & X & X & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & X & X & X & X & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & X & X & 0 & X & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & X & X & X & X & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & X & X & X & X & X \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & X & X & X & X \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & X & X & X \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & X & X \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & X
 \end{bmatrix}$$

FINALE



## Larghezza di banda (“bandwidth”)

X	X	X	X	0	0	0	0	0	0	0	0
X	X	X	X	0	0	0	0	0	0	0	0
X	X	X	X	X	X	0	0	0	0	0	0
X	X	X	X	0	X	X	0	0	0	0	0
0	0	X	0	X	X	X	X	0	0	0	0
0	0	X	X	X	X	X	0	X	0	0	0
0	0	0	X	X	X	X	X	X	0	0	0
0	0	0	0	X	0	X	X	X	X	X	0
0	0	0	0	0	X	X	X	X	X	X	X
0	0	0	0	0	0	0	X	X	X	X	X
0	0	0	0	0	0	0	0	X	X	X	X

$$N^{\circ} \text{ operazioni} \approx (\text{largh. banda})^2 \cdot n_{GDL}$$





Larghezza di banda

*dipende dal*

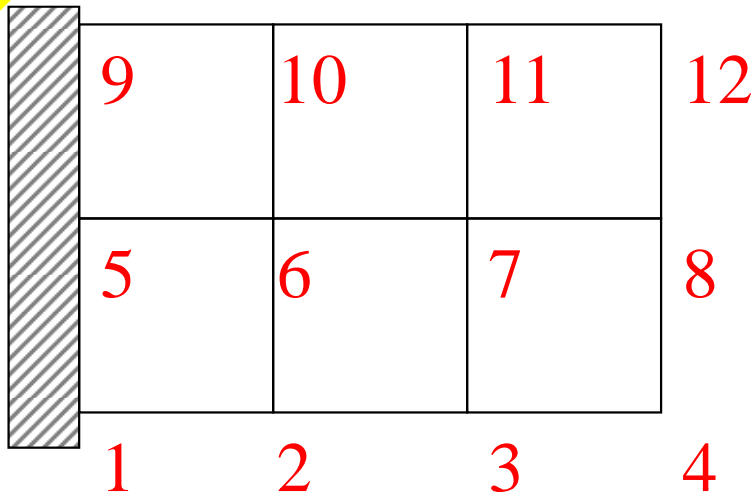
Modo di costruire [K]

Esistono due modi principali di costruire la matrice [K]:

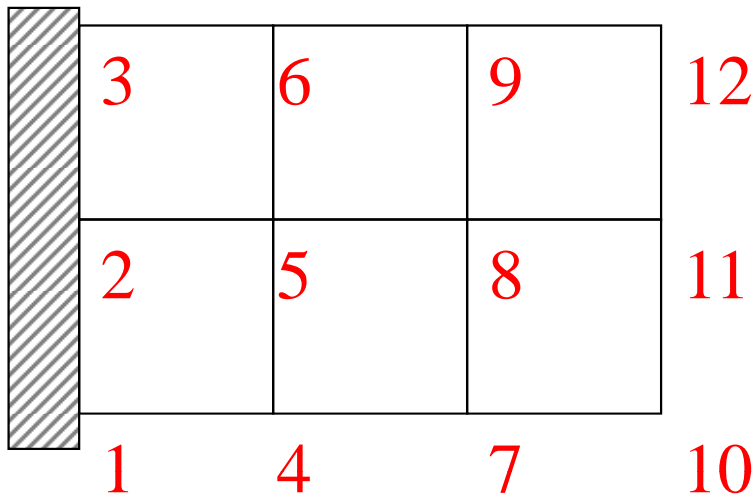
- seguendo l'ordine progressivo dei nodi;
- seguendo l'ordine progressivo degli elementi



# ORDINE NODI



Largh. Banda = 12



Largh. Banda = 10

Max. diff. n° d'ordine per nodi attaccati allo stesso elemento

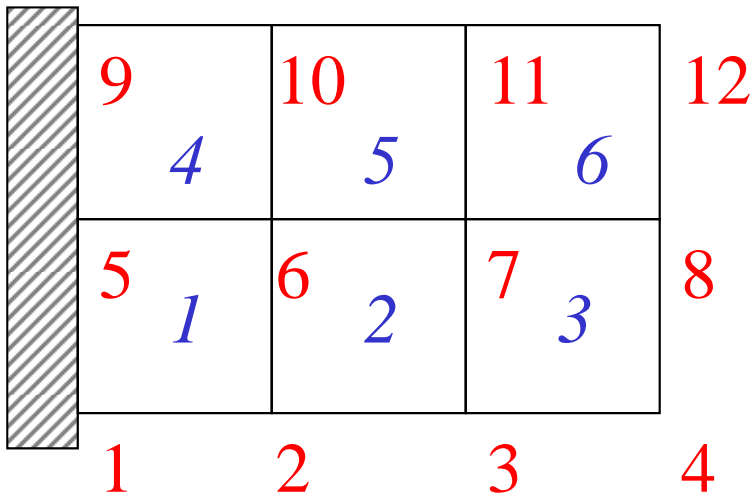
$N^\circ$  g.d.l. per nodo

$$\text{Largh. banda} = (n_{nE} + 1)n_{GDL,n}$$

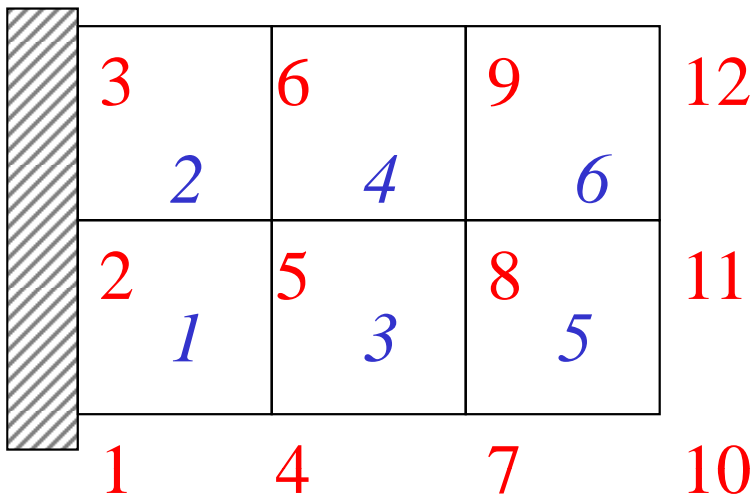
	1	2	3	4	5	6	7	8	9	10	11	12
X	X	0	0	X	X	0	0	0	0	0	0	0
	X	X	0	X	X	X	0	0	0	0	0	0
		X	X	0	X	X	X	0	0	0	0	0
			X	0	0	X	X	0	0	0	0	0
				X	X	0	0	X	X	0	0	0
					X	X	0	X	X	X	0	0
						X	X	0	X	X	X	0
							X	X	0	X	X	0
								X	0	0	X	X
									X	X	0	0
										X	X	0
											X	X
												X



## ORDINE ELEMENTI



Largh. Banda = 16

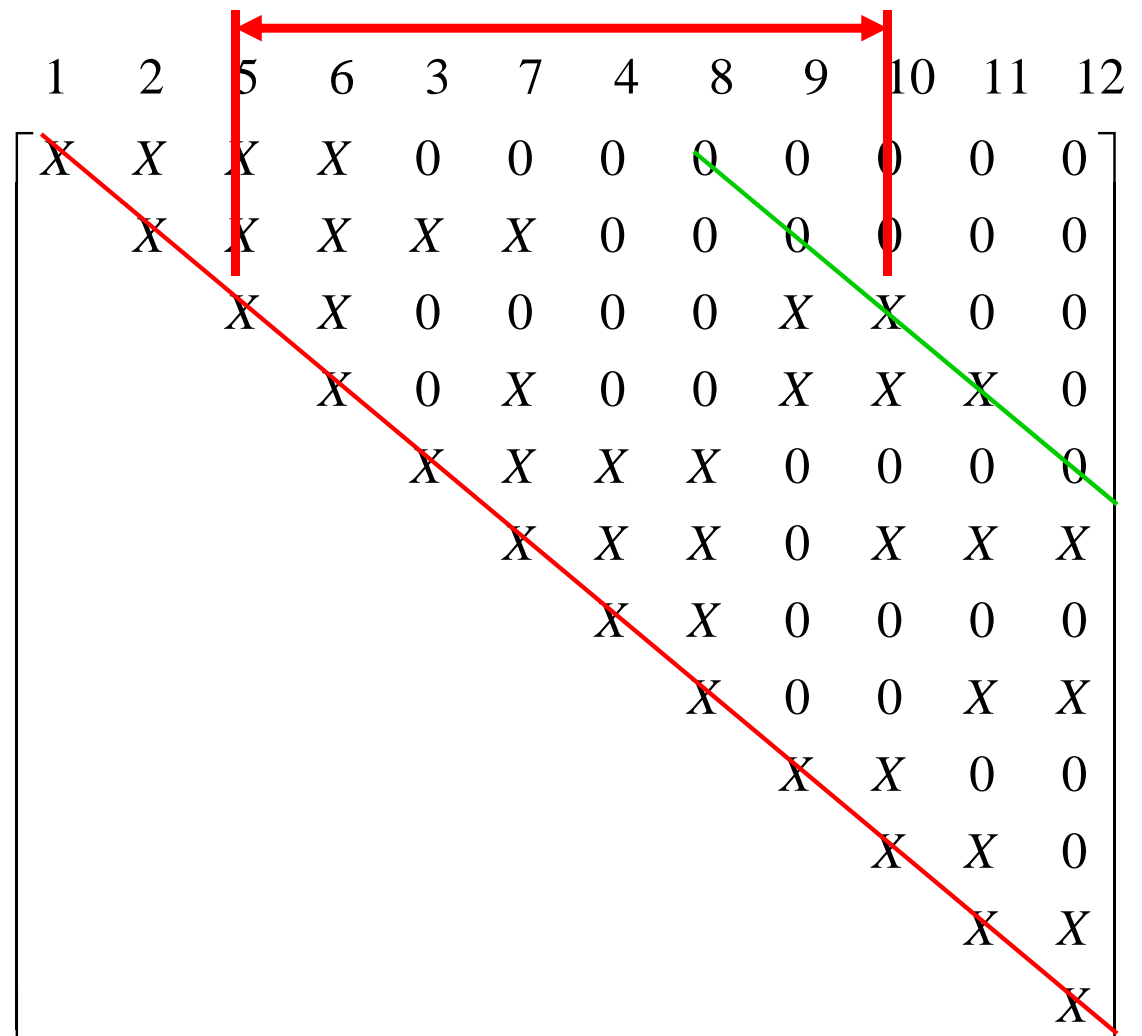


Largh. Banda = 12

Max. diff. n° d'ordine per elementi attaccati allo stesso nodo

$N^\circ$  nodi per elemento/2

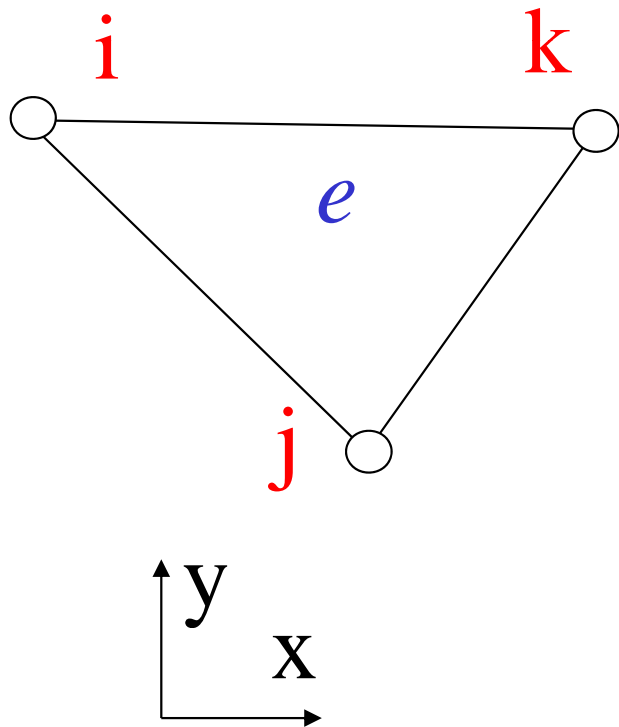
Largh. banda  $\sim (n_{En})n_{nod,e}n_{GDL,n}$





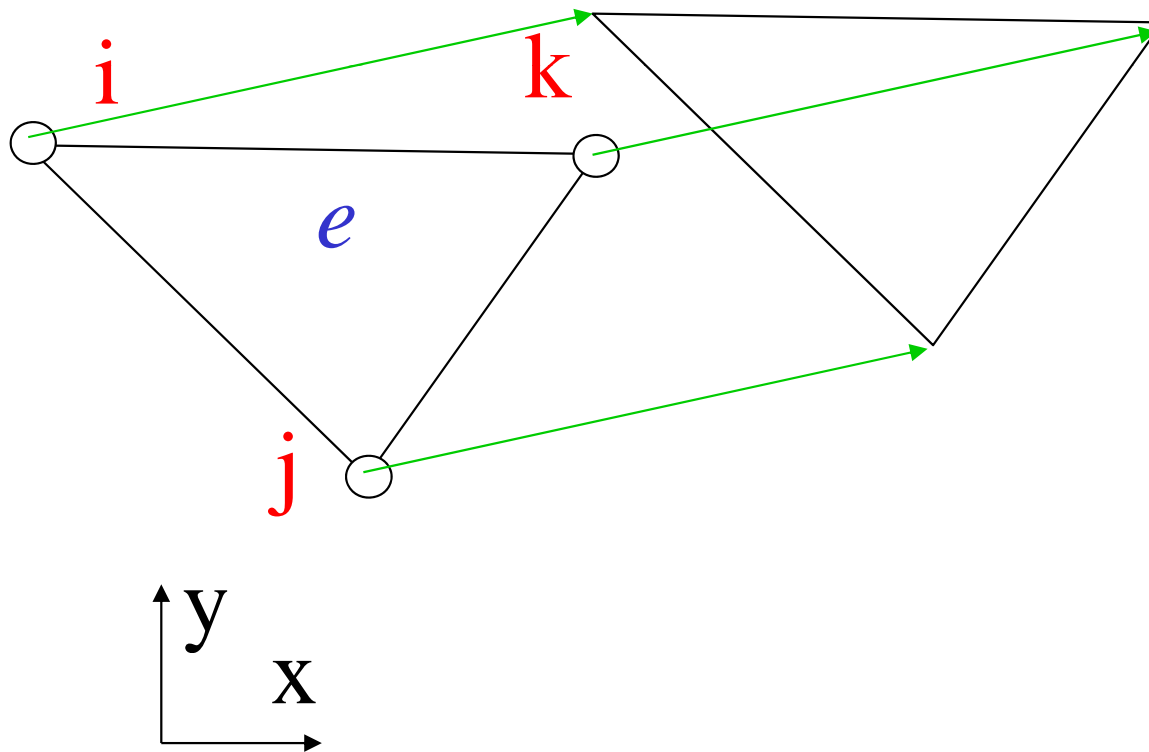
# Condizioni di convergenza sulle funz.ni di forma

**Condizione 1:** la f.ne di spostamento deve dare luogo ad una deformazione nulla in tutti i punti dell'elemento quando il campo di spostamenti nodali corrisponde ad un moto rigido.



# Condizioni di convergenza sulle funz.ni di forma

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## Verifica per elemento triangolare

$$\{U^e\} = \begin{Bmatrix} u_x \\ u_y \\ u_x \\ u_y \\ u_x \\ u_y \end{Bmatrix}$$



Verifica per elemento triangolare

$$\{U^e\} = \begin{Bmatrix} u_x \\ u_y \\ u_x \\ u_y \\ u_x \\ u_y \end{Bmatrix}$$

$$\{\varepsilon\} = [B]\{U^e\}$$



Verifica per elemento triangolare

$$\{U^e\} = \begin{Bmatrix} u_x \\ u_y \\ u_x \\ u_y \\ u_x \\ u_y \end{Bmatrix}$$

$$[B] = \begin{bmatrix} B_{11} & 0 & B_{13} & 0 & B_{15} & 0 \\ 0 & C_{22} & 0 & C_{24} & 0 & C_{26} \\ C_{11} & B_{22} & C_{13} & B_{24} & C_{15} & B_{26} \end{bmatrix}$$

$$\{\varepsilon\} = [B]\{U^e\}$$





Verifica per elemento triangolare

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$$\{\varepsilon\} = [B]\{U^e\}$$

$$\varepsilon_x = B_{11}u_x + B_{13}u_x + B_{15}u_x$$



Verifica per elemento triangolare

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$$\{\varepsilon\} = [B]\{U^e\}$$

$$\varepsilon_x = B_{11}u_x + B_{13}u_x + B_{15}u_x$$

$$B_{11} = \frac{y_j - y_k}{2\Delta}$$

$$B_{13} = \frac{y_k - y_i}{2\Delta}$$

$$B_{15} = \frac{y_i - y_j}{2\Delta}$$



Verifica per elemento triangolare

$$\{U^e\} = \begin{Bmatrix} u_x \\ u_y \\ u_x \\ u_y \\ u_x \\ u_y \end{Bmatrix}$$

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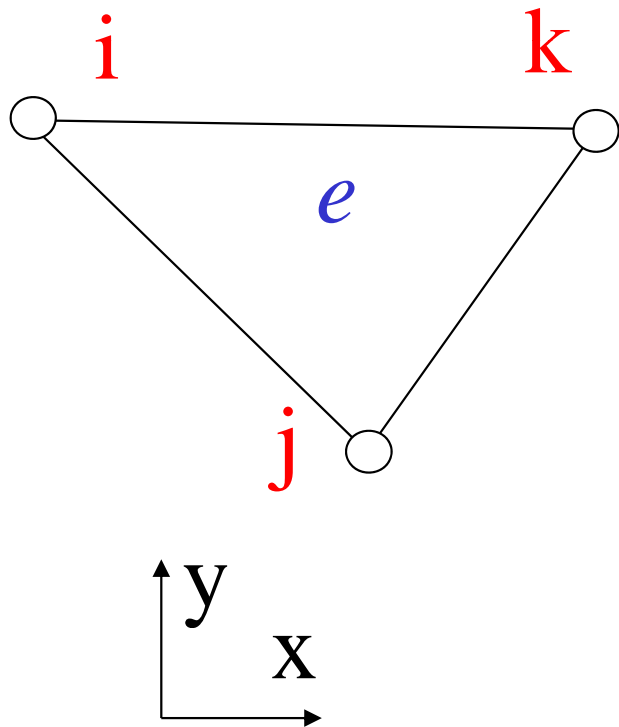
$$B_{11} = \frac{y_j - y_k}{2\Delta}$$

$$B_{13} = \frac{y_k - y_i}{2\Delta}$$

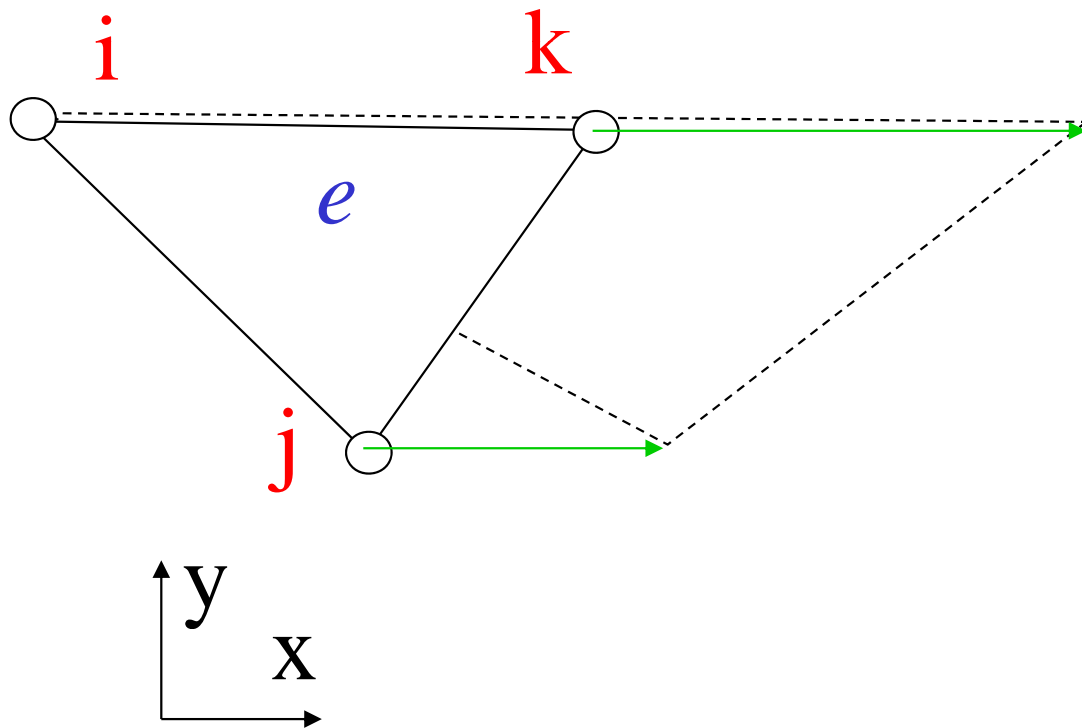
$$B_{15} = \frac{y_i - y_j}{2\Delta}$$

$$\varepsilon_x = \frac{y_j - y_k}{2\Delta}u_x + \frac{y_k - y_i}{2\Delta}u_x + \frac{y_i - y_j}{2\Delta}u_x = 0$$

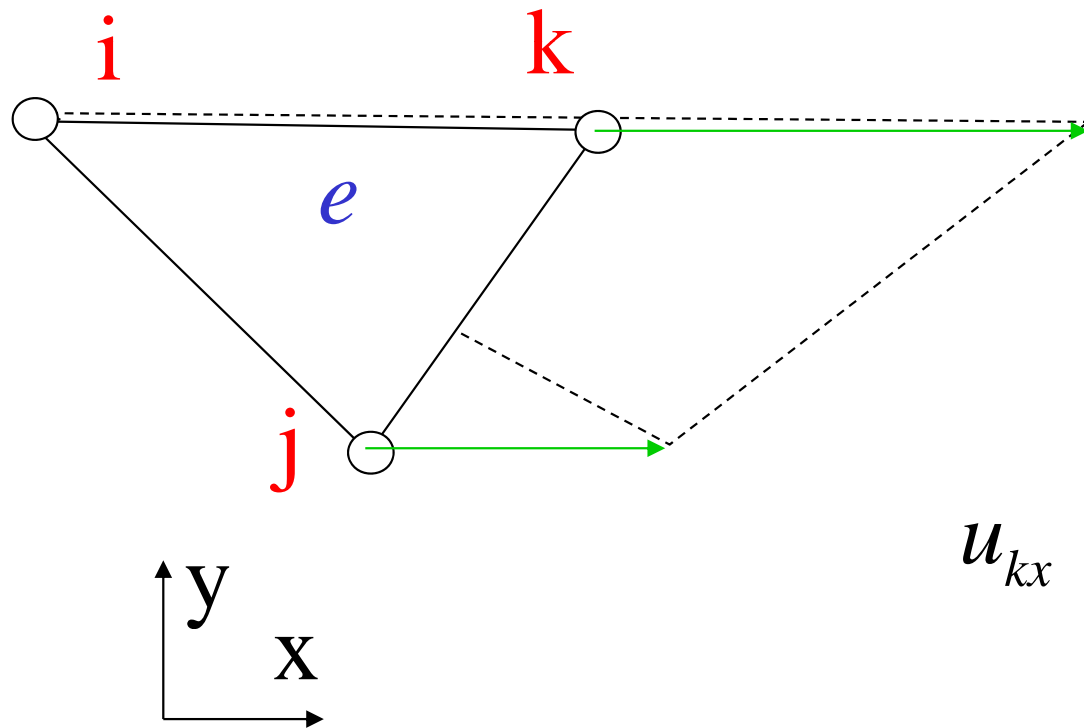
**Condizione 2:** la f.ne di spostamento deve dare luogo ad una deformazione costante in tutti i punti dell'elemento quando il campo di spostamenti nodali è compatibile con tale condizione.



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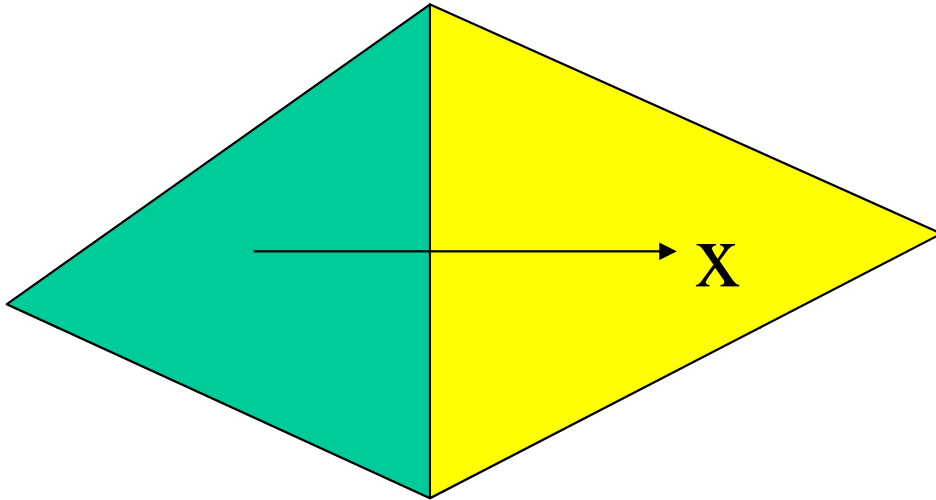
**Condizione 2:** la f.ne di spostamento deve dare luogo ad una deformazione costante in tutti i punti dell'elemento quando il campo di spostamenti nodali è compatibile con tale condizione.



$$u_{kx} = \frac{(x_k - x_i)}{(x_j - x_i)} u_{jx}$$

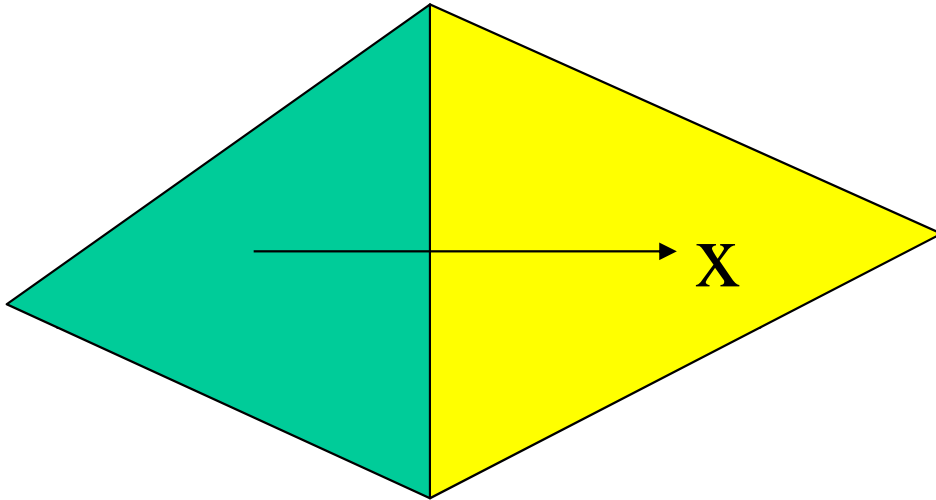


**Condizione 3:** la f.ne di spostamento deve dare luogo a deformazioni limitate all'interfaccia tra elementi diversi.





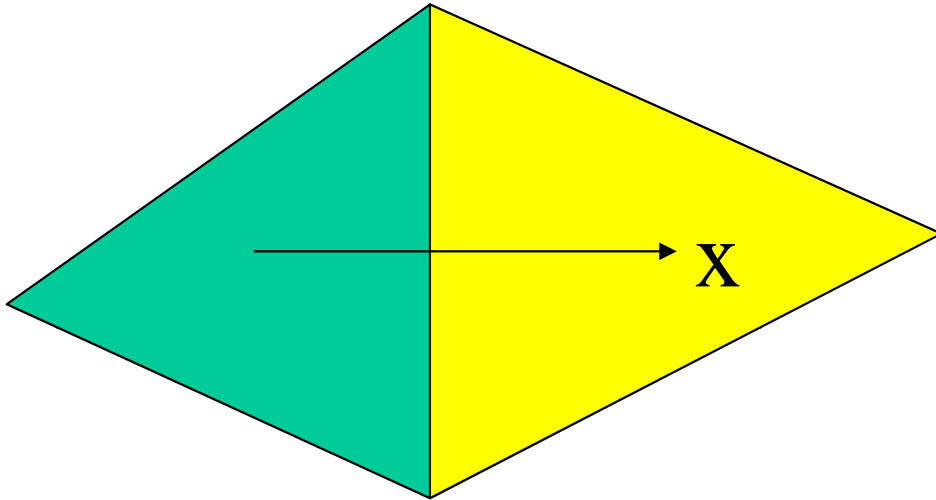
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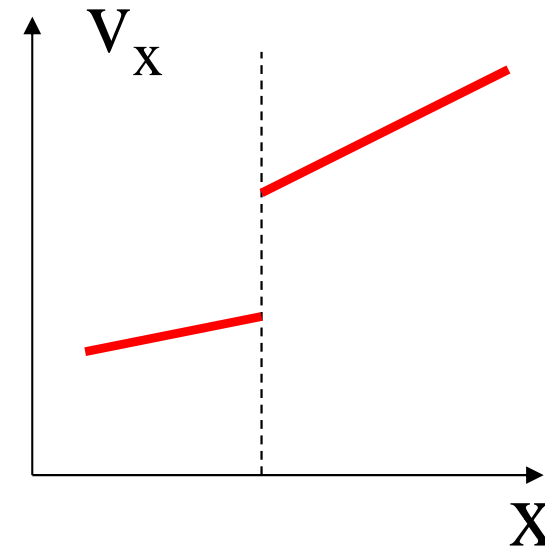
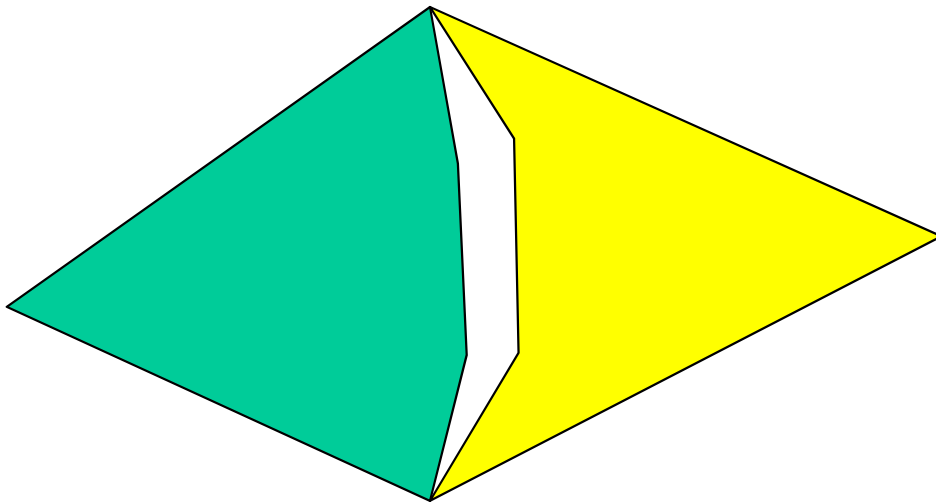
$$\epsilon_x = \frac{\partial v_x}{\partial x}$$



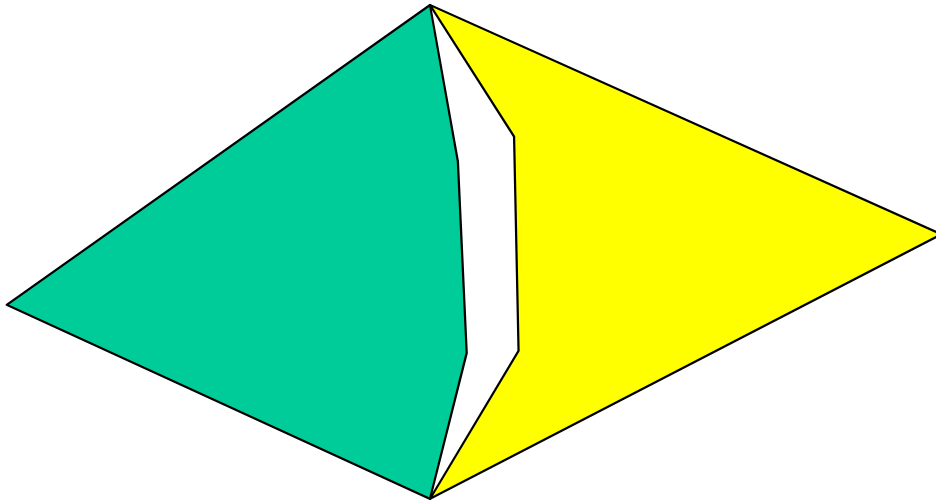
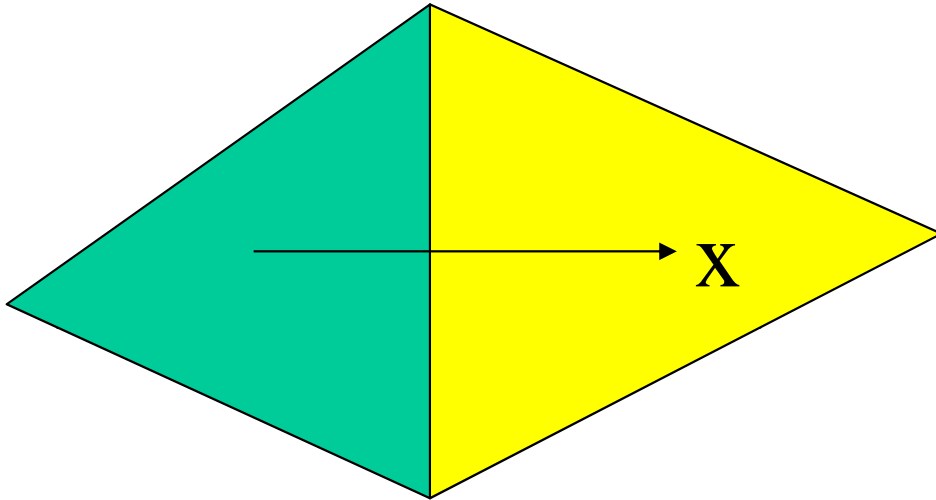
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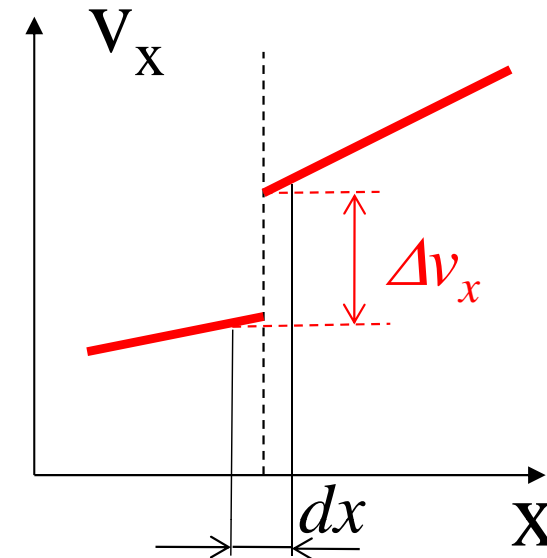
$$\varepsilon_x = \frac{\partial v_x}{\partial x}$$



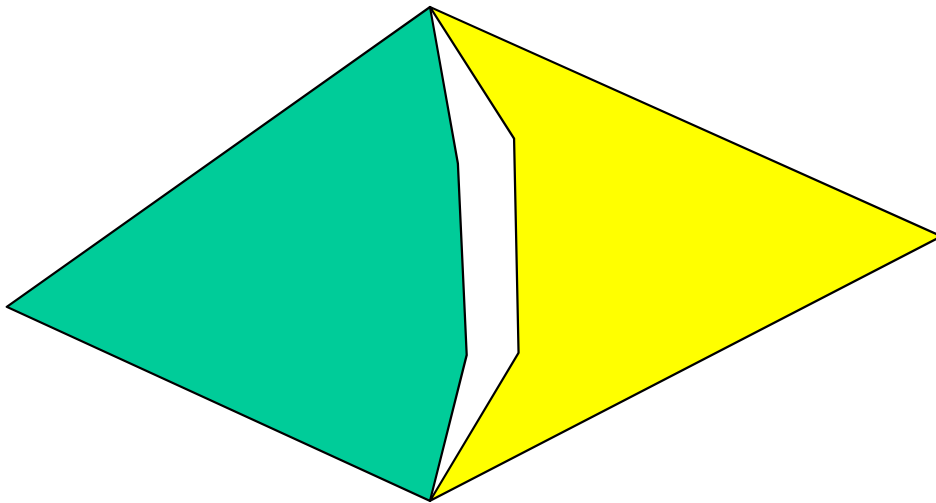
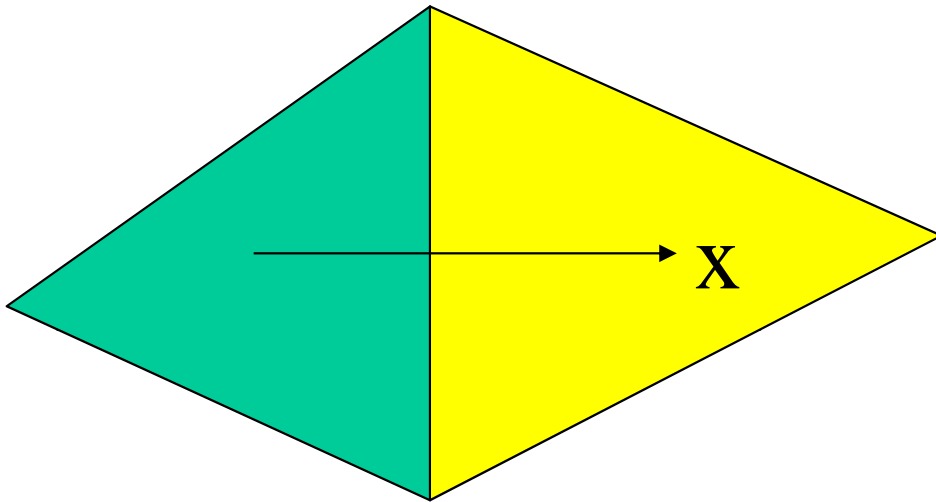
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$$\varepsilon_x = \frac{\partial v_x}{\partial x}$$

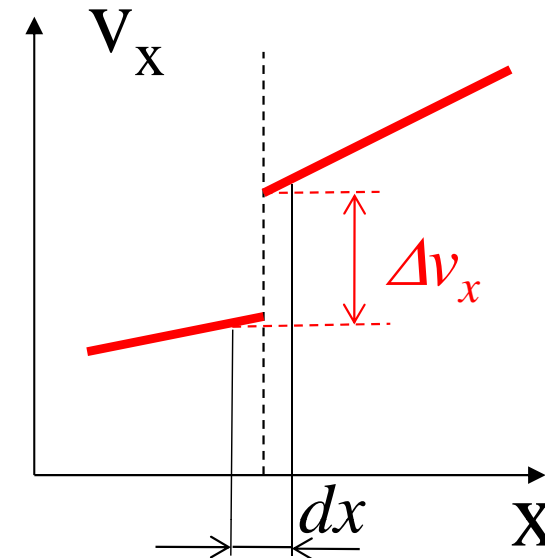


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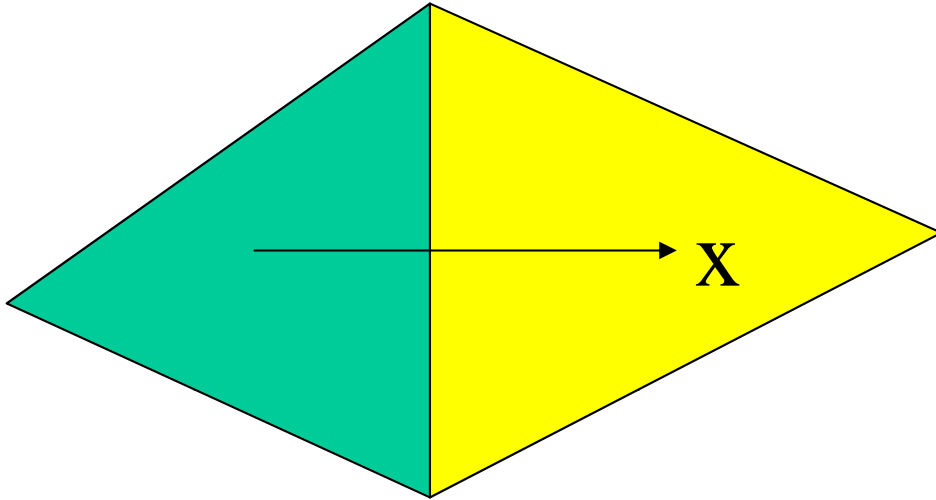
$\varepsilon \rightarrow \infty$

$$\varepsilon_x = \frac{\partial v_x}{\partial x}$$



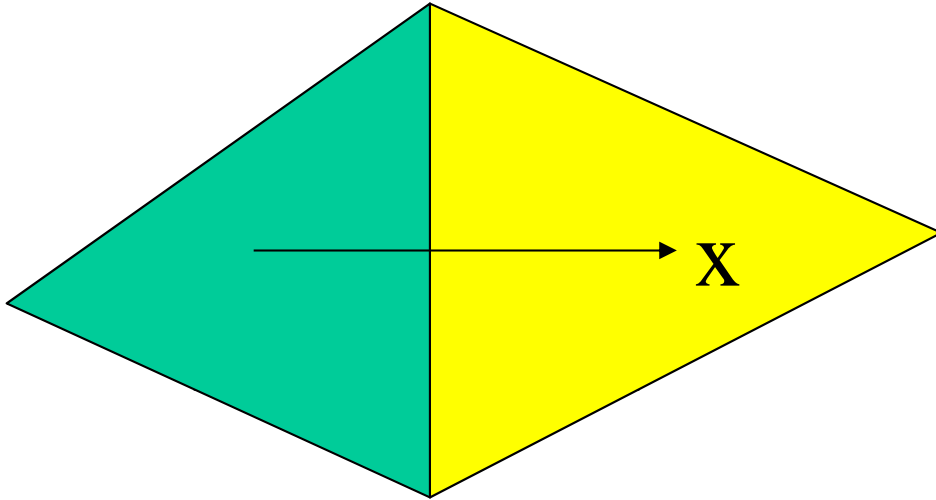


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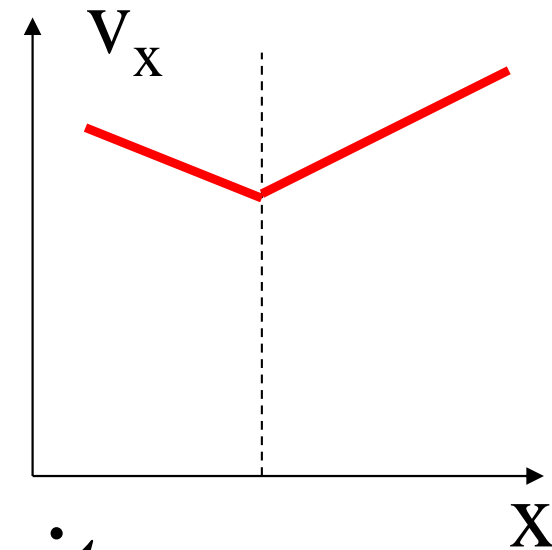
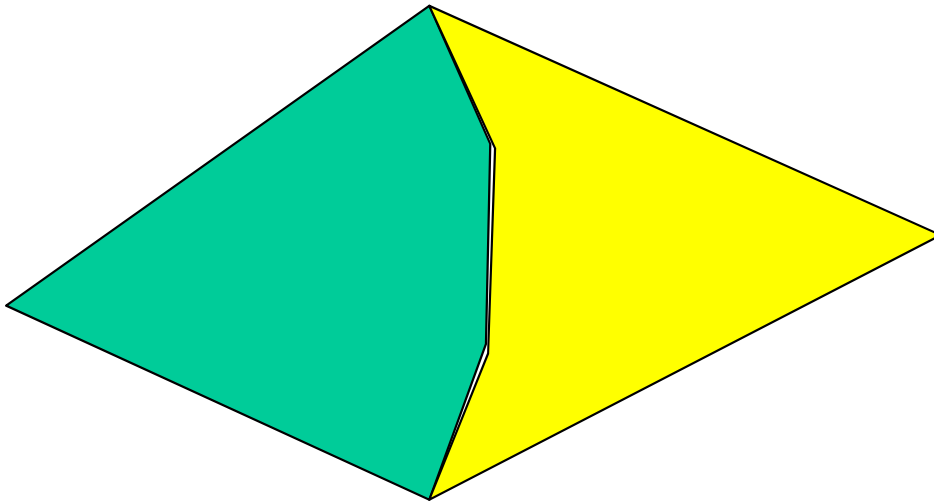


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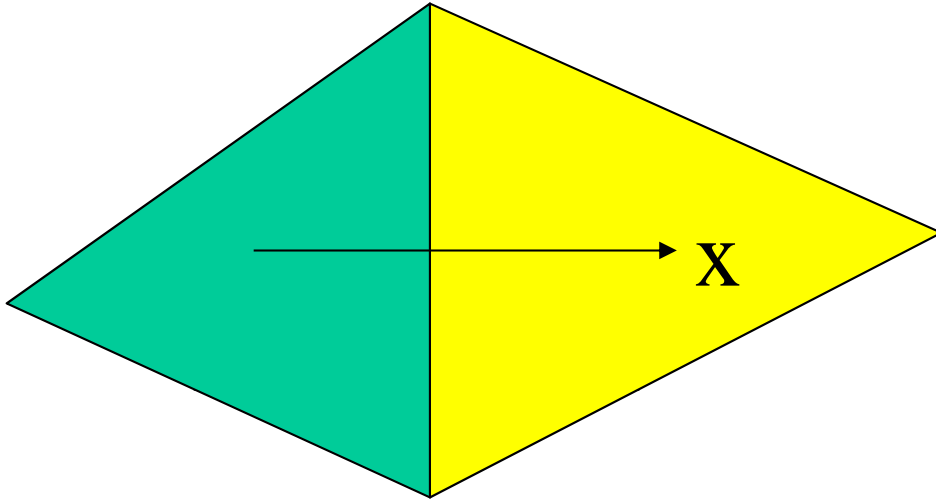
$$\varepsilon_x = \frac{\partial v_x}{\partial x}$$



$\varepsilon \rightarrow$  *valore finito*



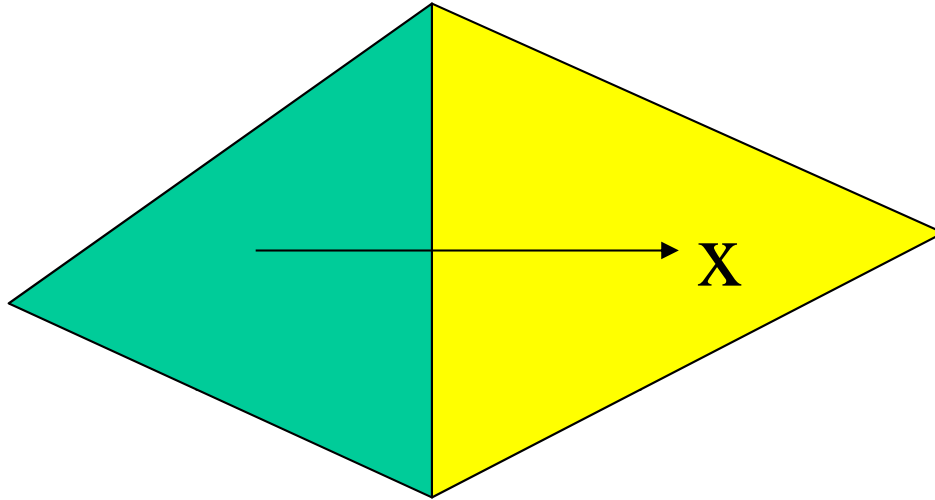
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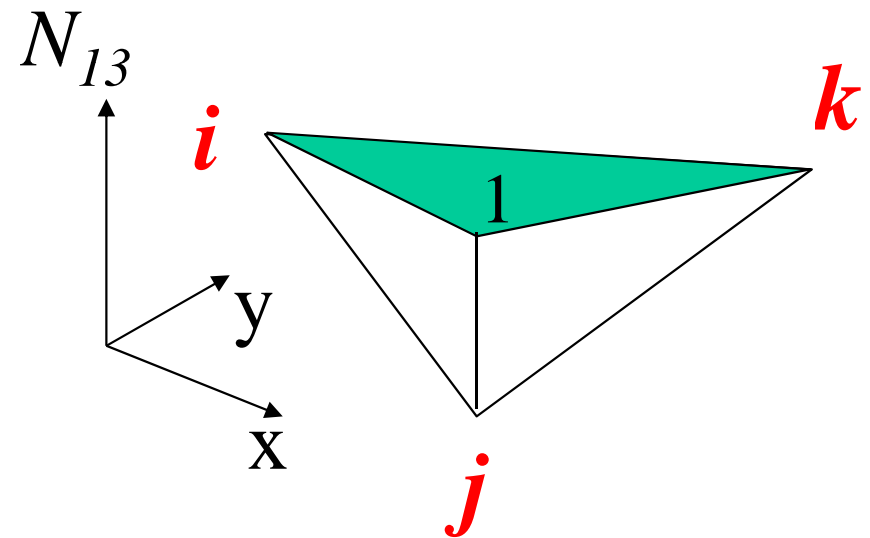
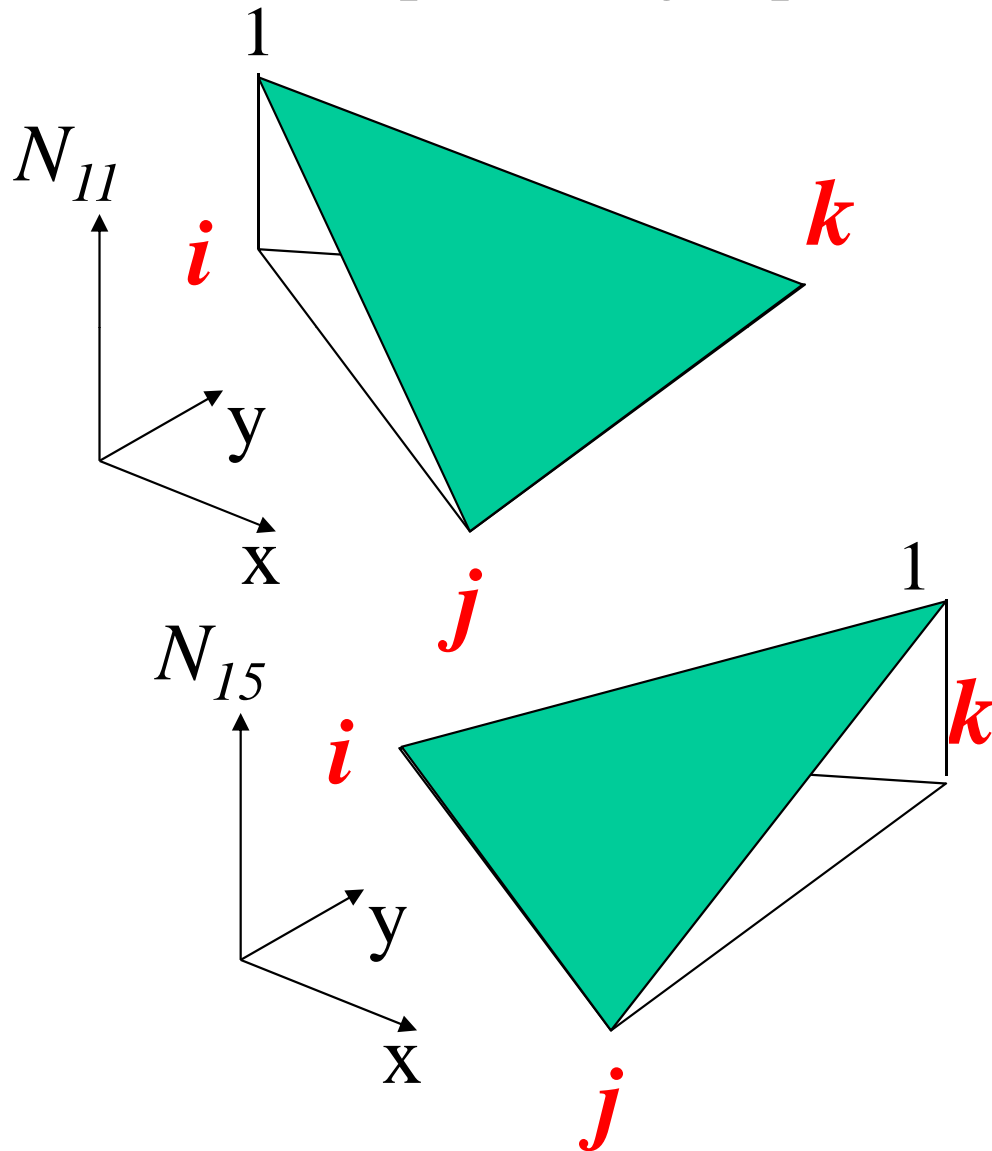


$$\varepsilon_x = \frac{\partial v_x}{\partial x}$$

In generale:

Se le  $\varepsilon$  implicano la derivata *n-sima* della f.ne di spostamento, quest'ultima deve essere continua all'interfaccia con Classe di continuità  $C_{n-1}$

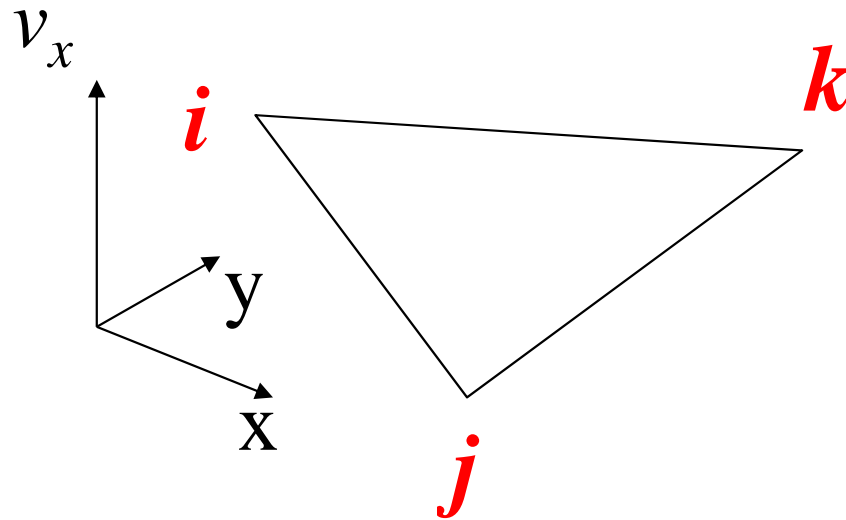
Oss.ne: la funzione di spostamento scelta garantisce tale continuità in quanto lo spostamento di un punto appartenente ad un lato non dipende dagli spostamenti del nodo opposto



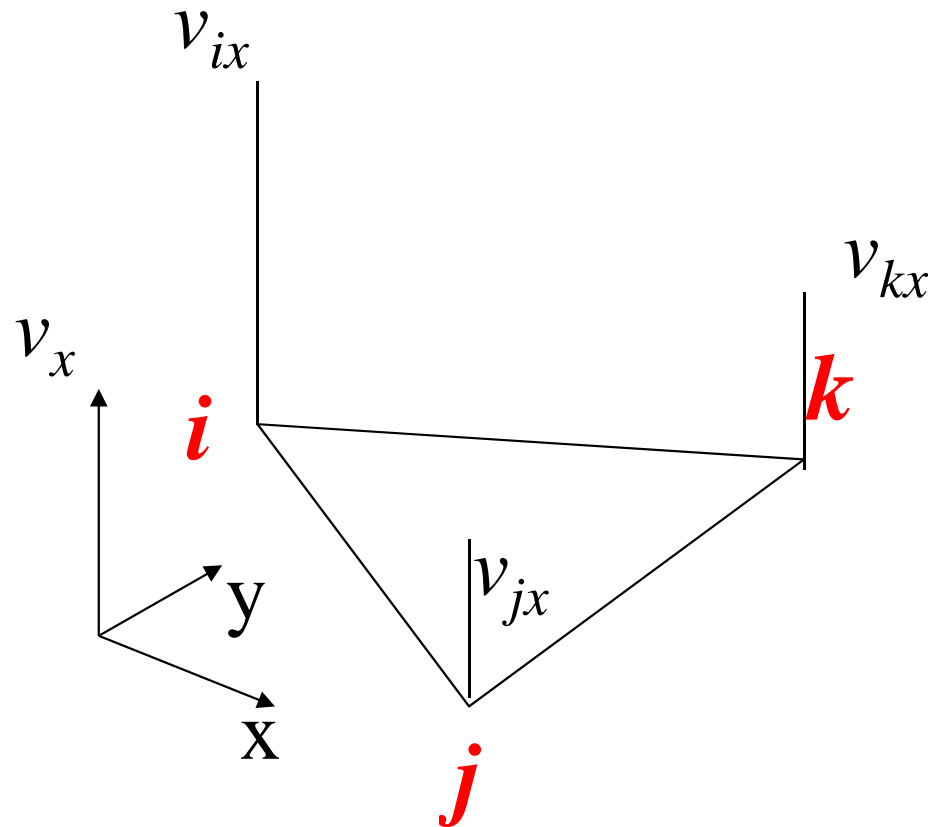




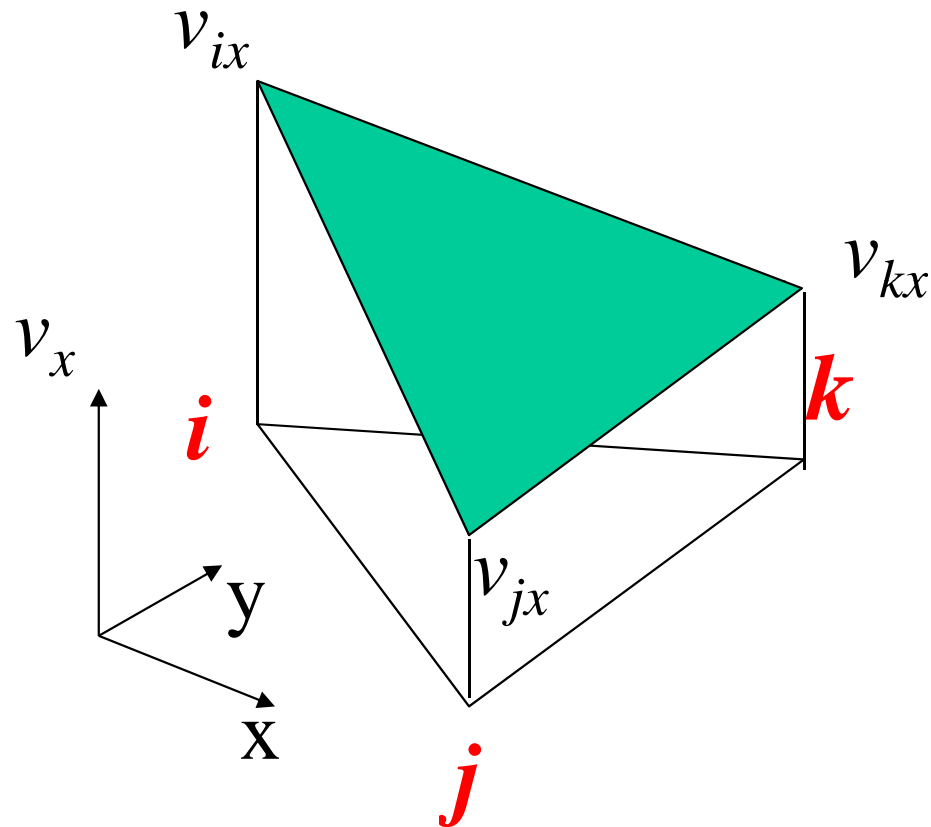
# Approssimazione effettiva del campo di spostamenti sul singolo elemento



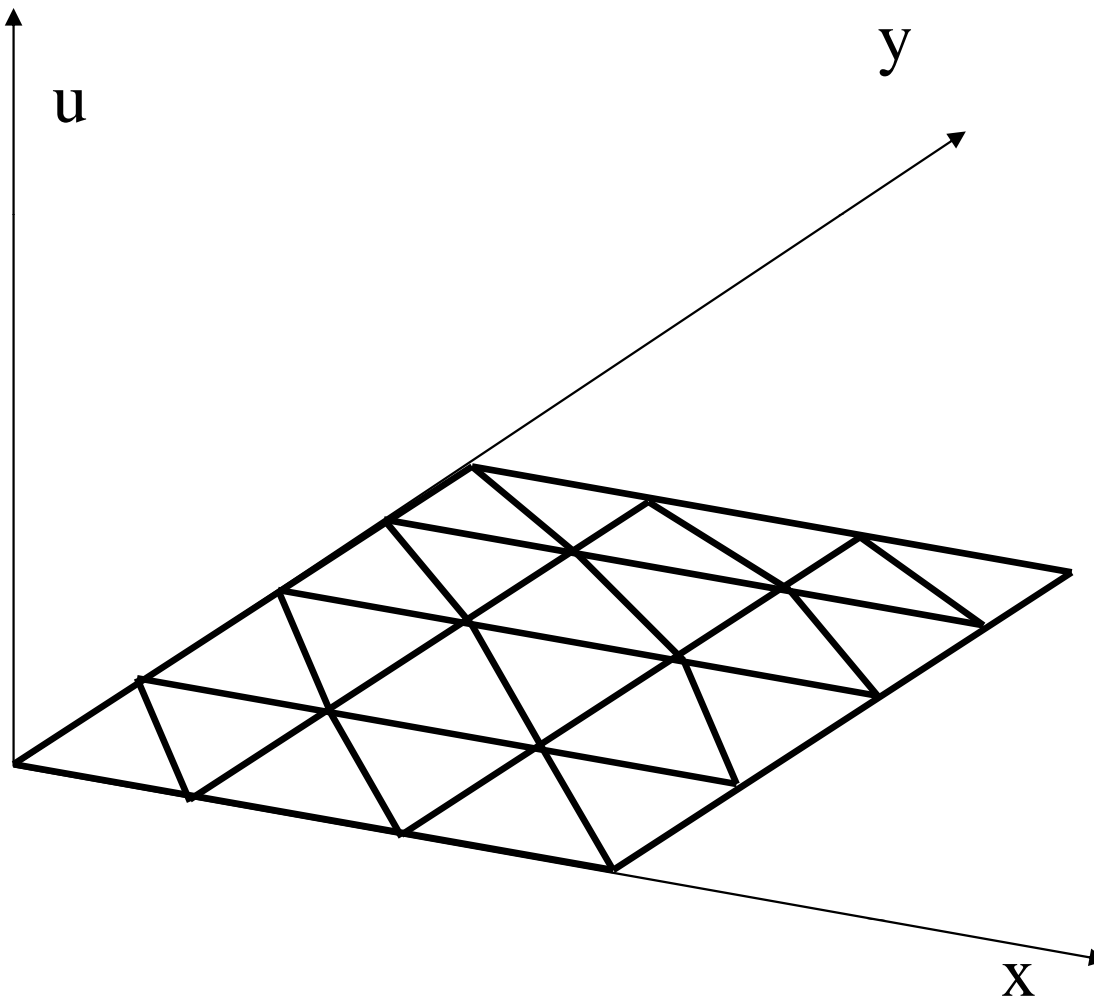
## Approssimazione effettiva del campo di spostamenti sul singolo elemento



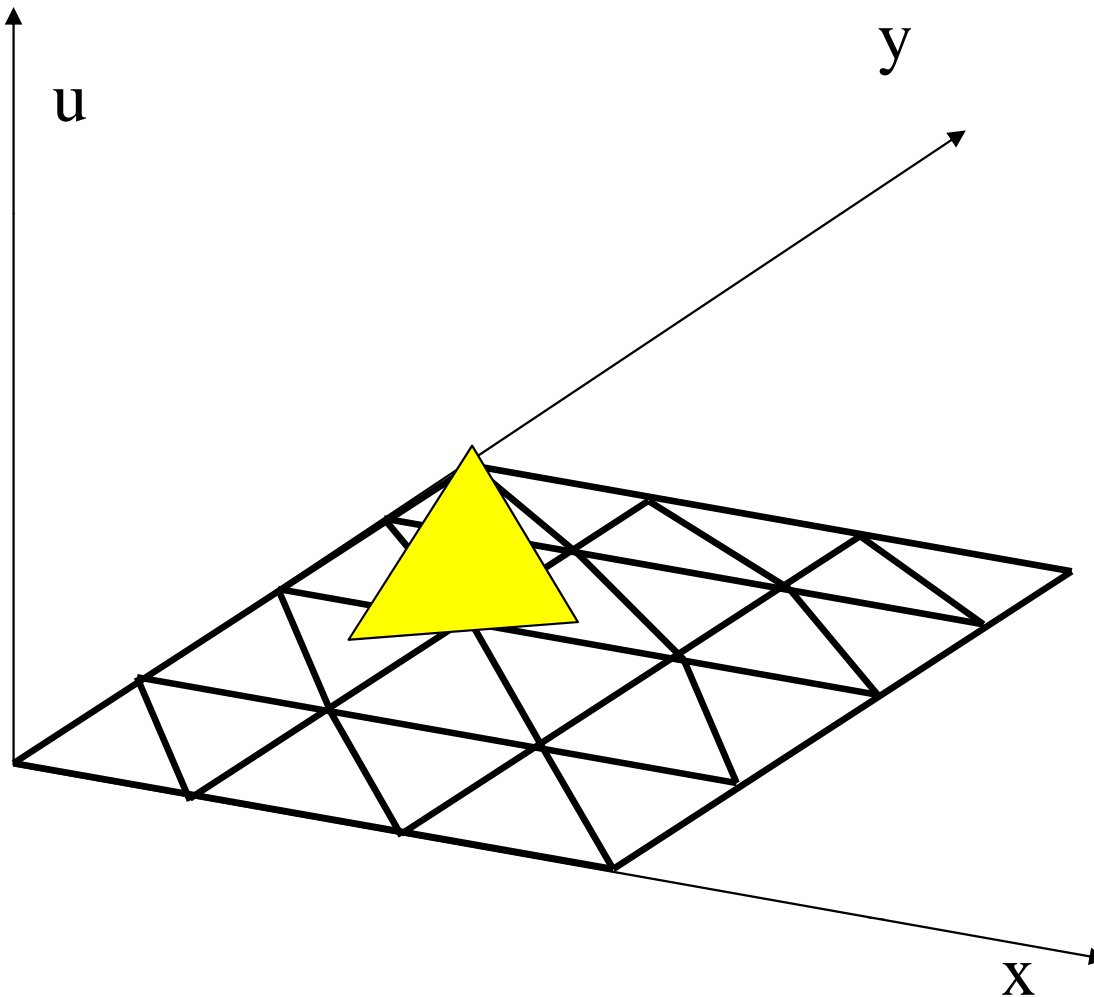
# Approssimazione effettiva del campo di spostamenti sul singolo elemento



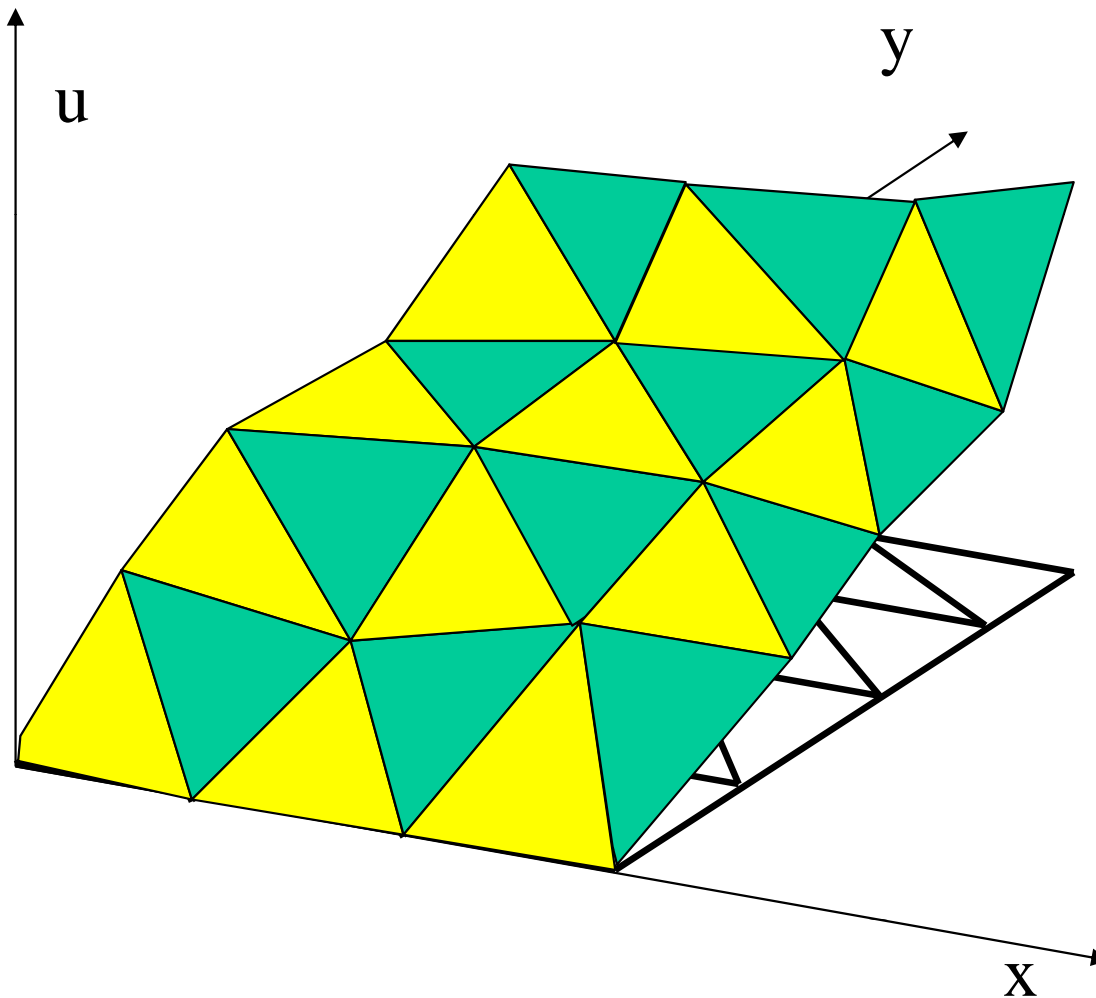
## Approssimazione effettiva del campo di spostamenti sull'intero modello



## Approssimazione effettiva del campo di spostamenti sull'intero modello

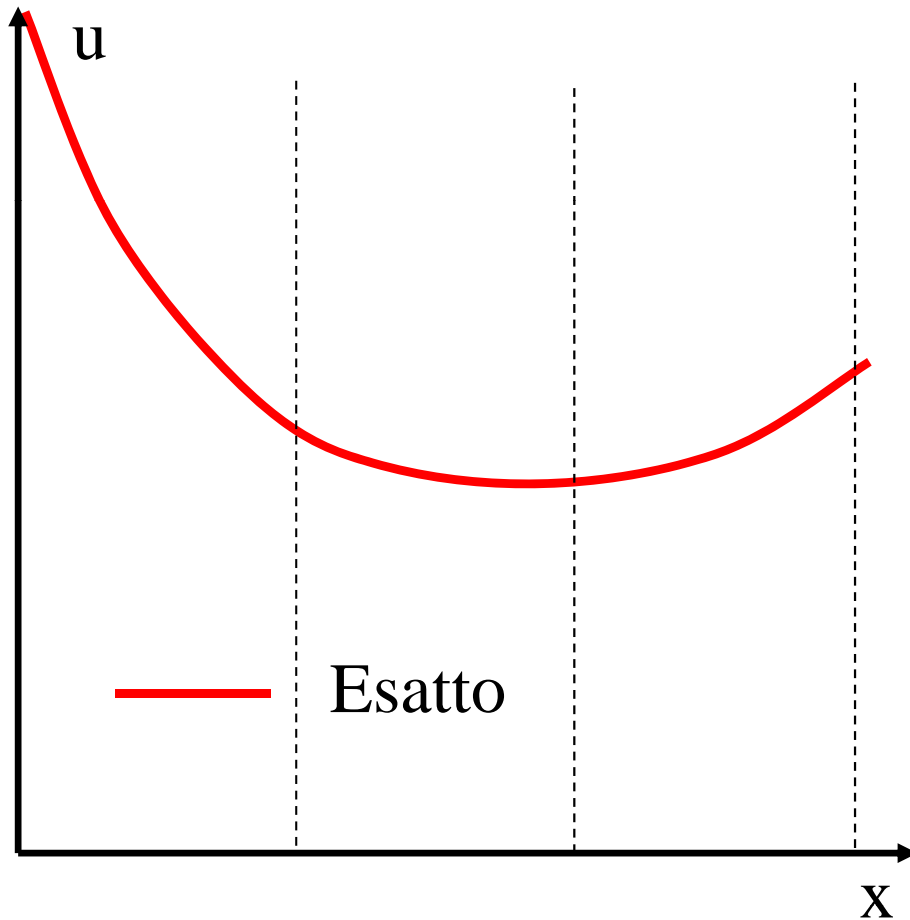
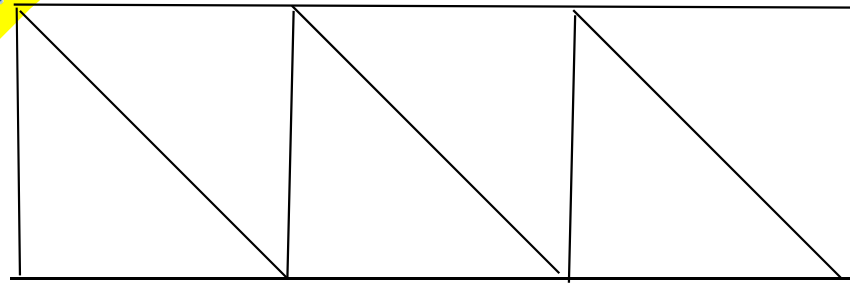


## Approssimazione effettiva del campo di spostamenti sull'intero modello



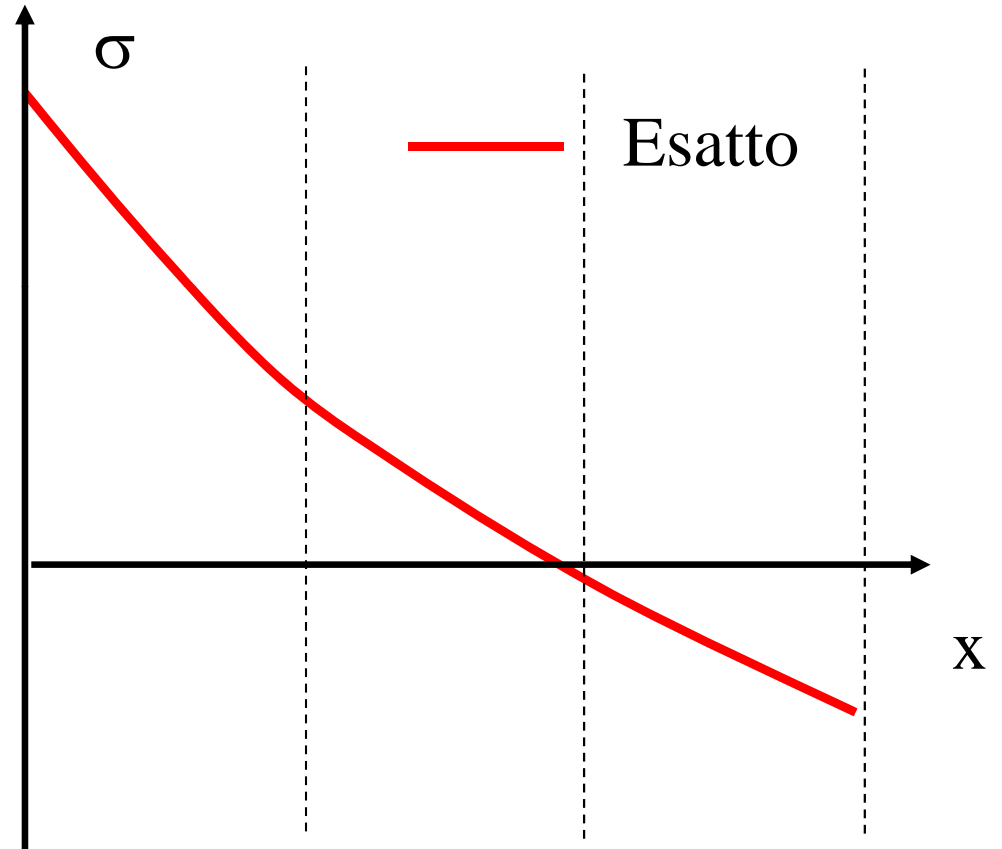
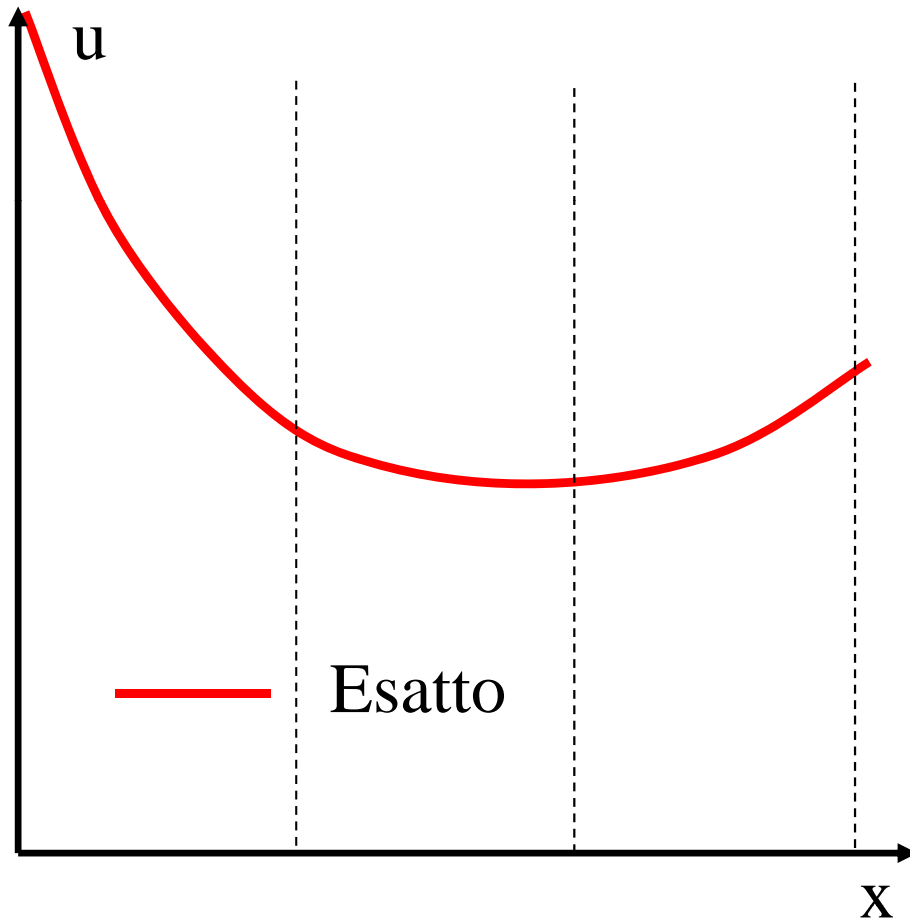
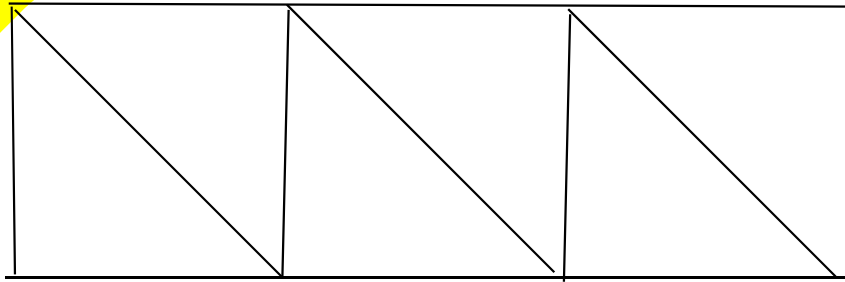


# Andamento effettivo delle tensioni





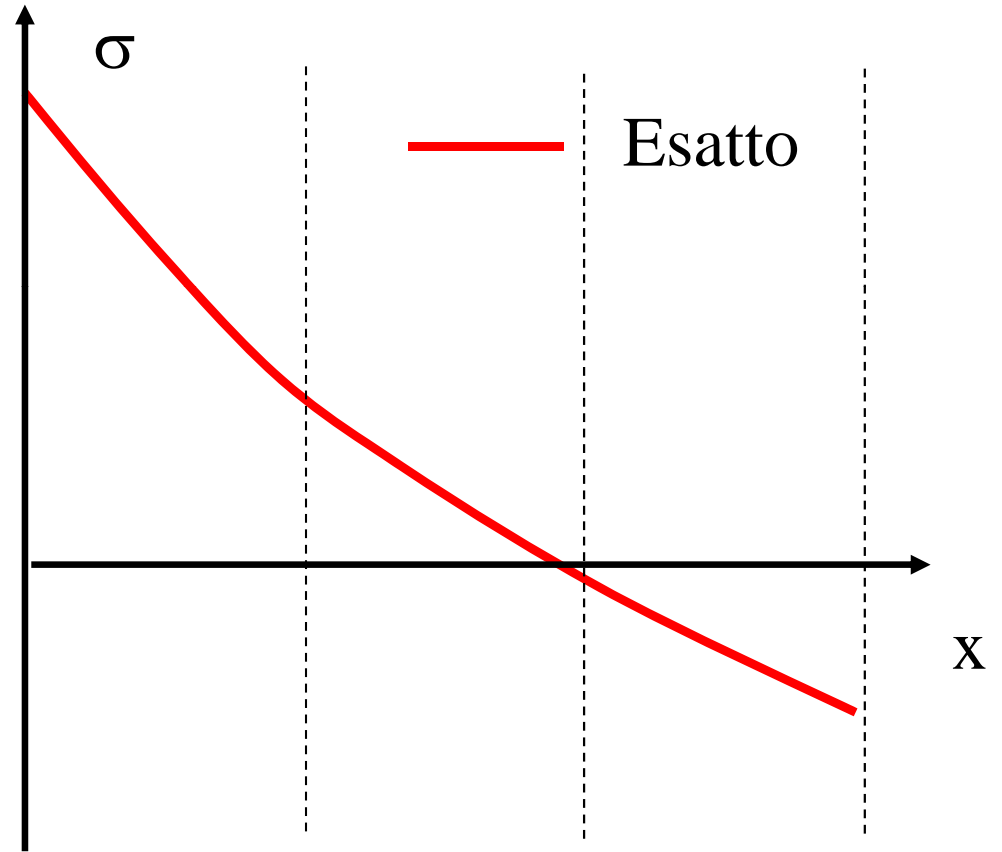
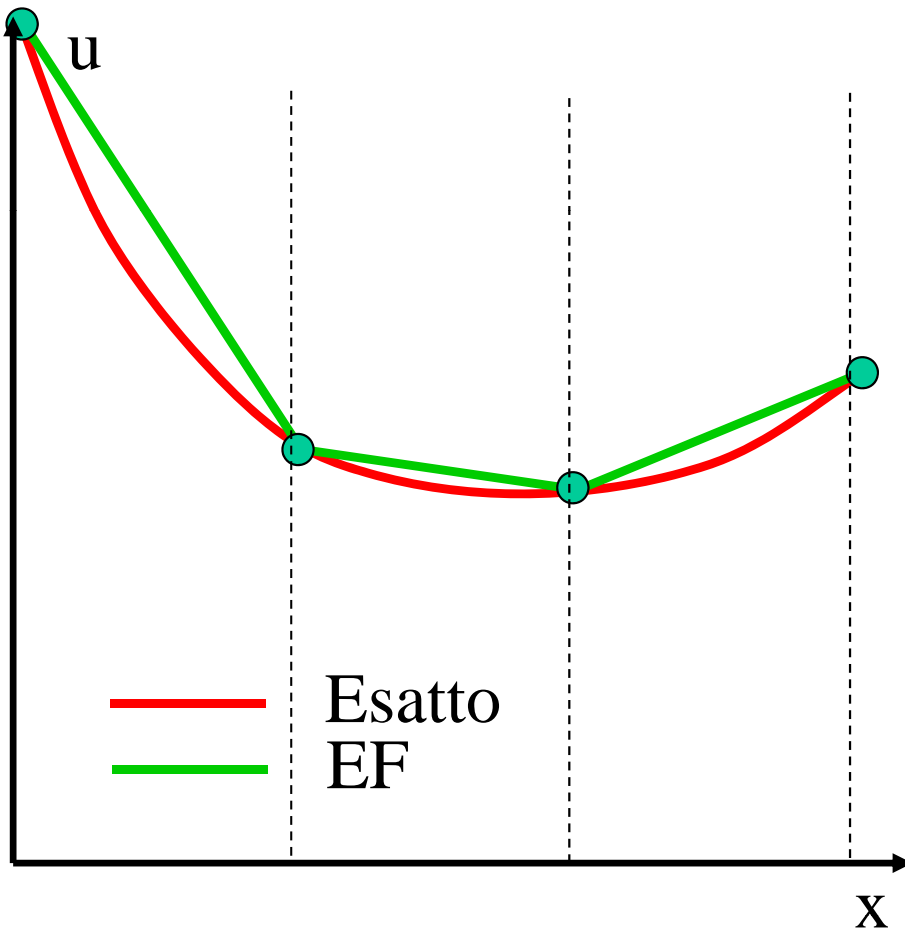
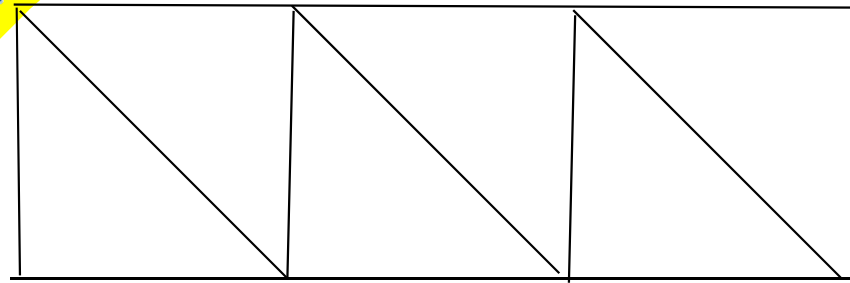
# Andamento effettivo delle tensioni





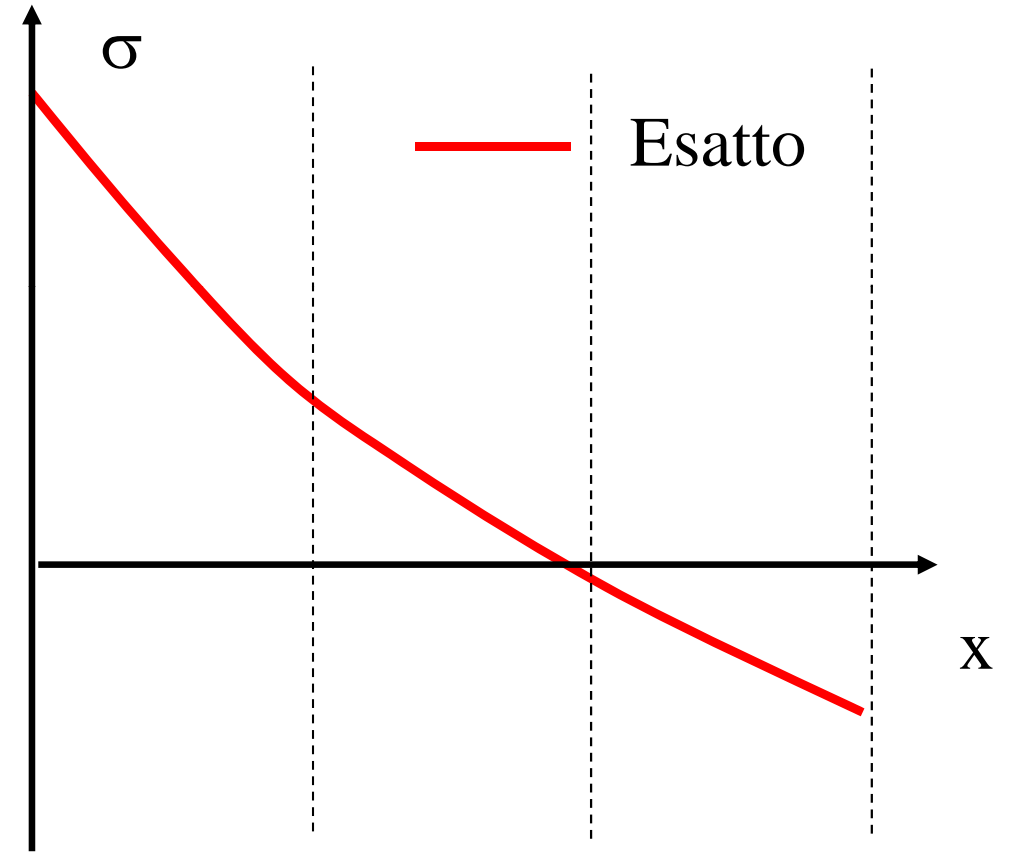
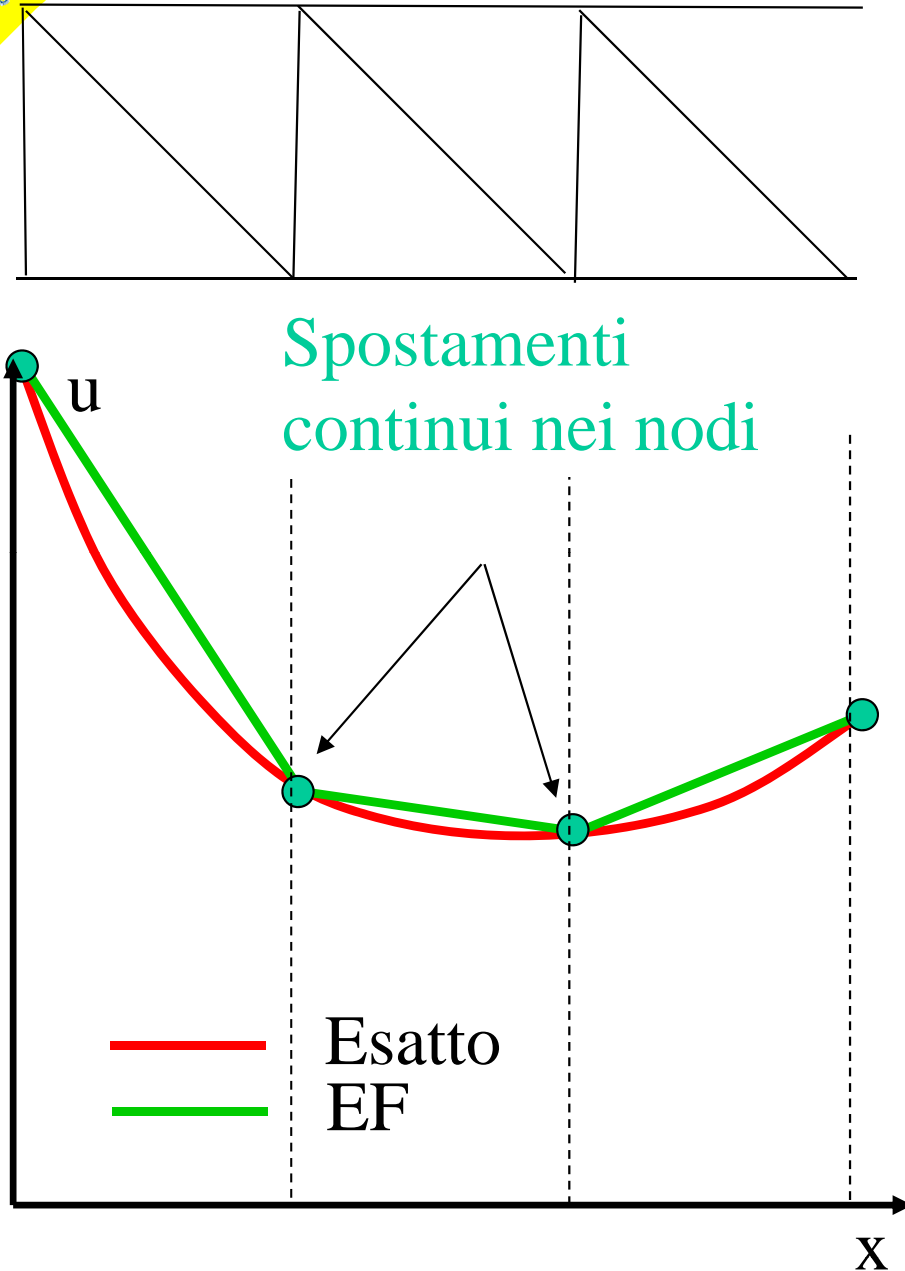


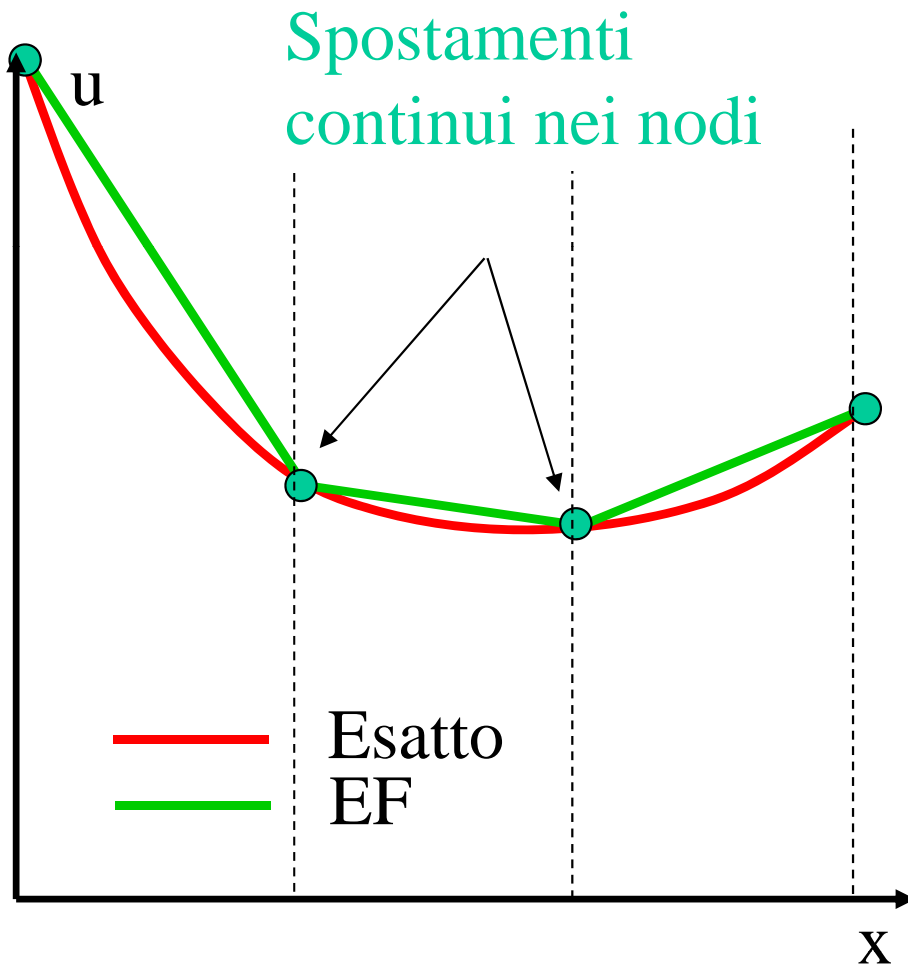
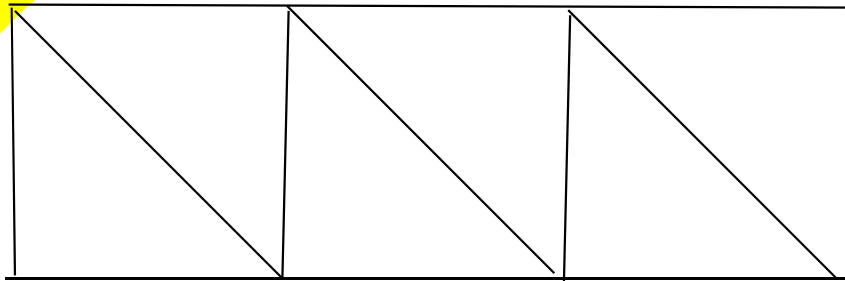
# Andamento effettivo delle tensioni





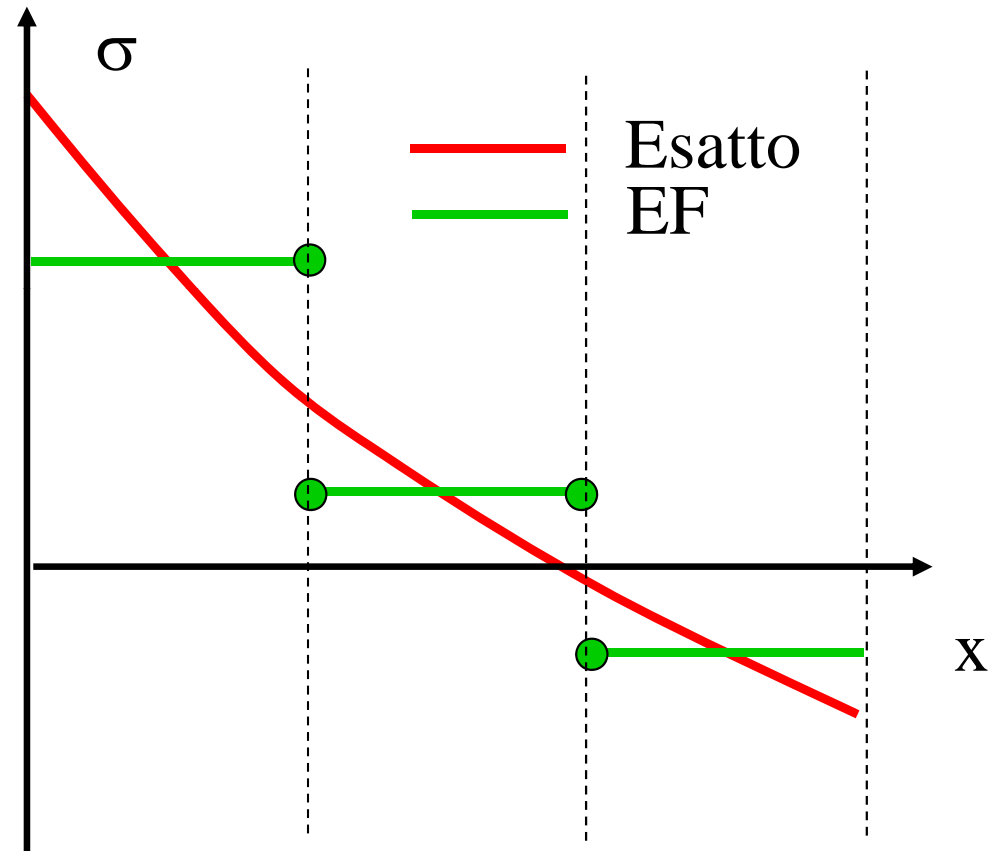
# Andamento effettivo delle tensioni

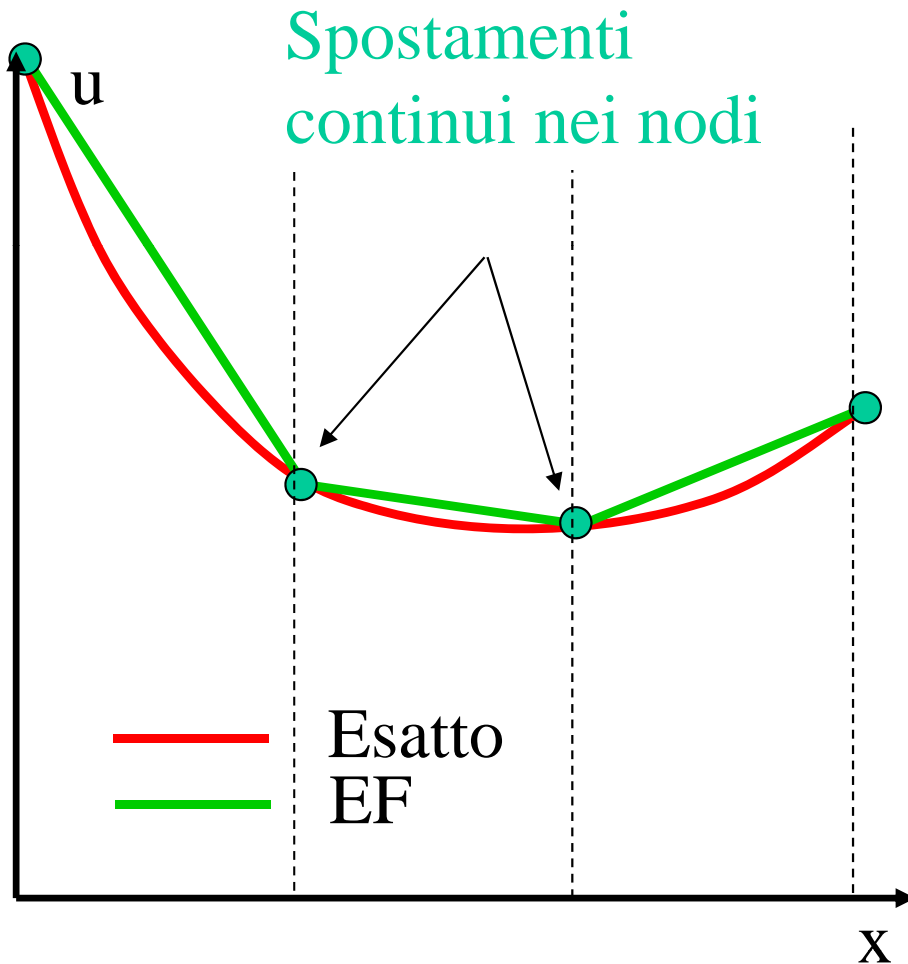
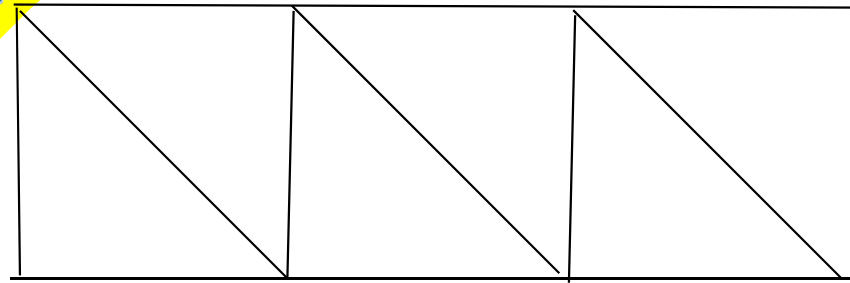




## Andamento effettivo delle tensioni

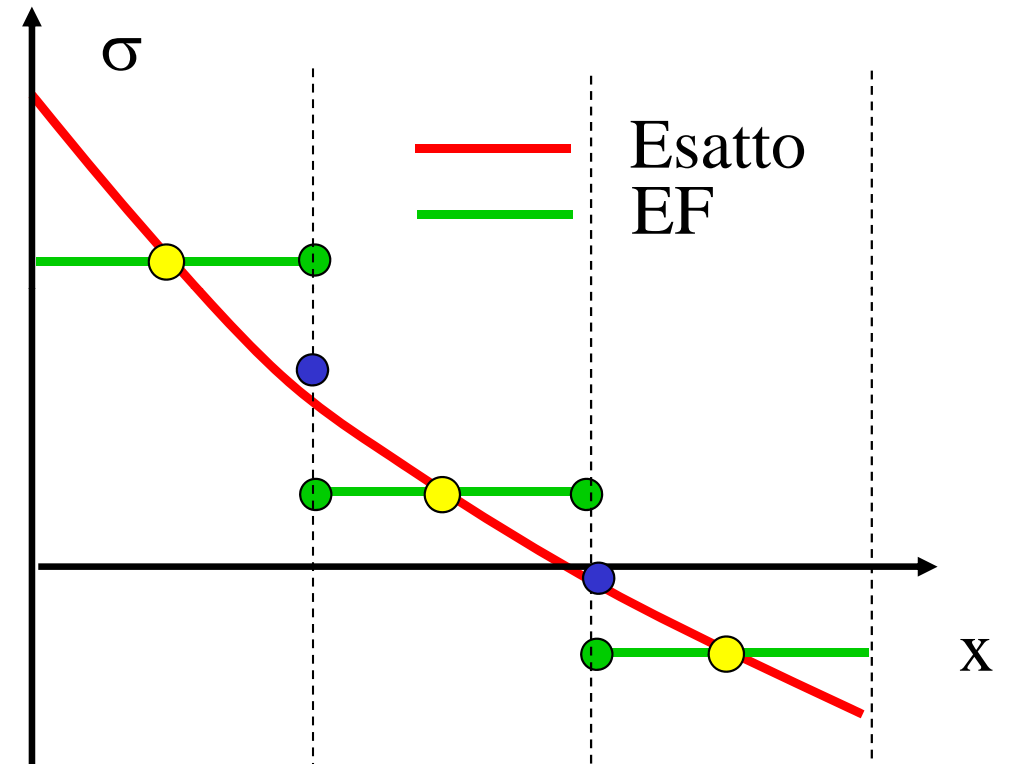
### Tensioni discontinue nei nodi



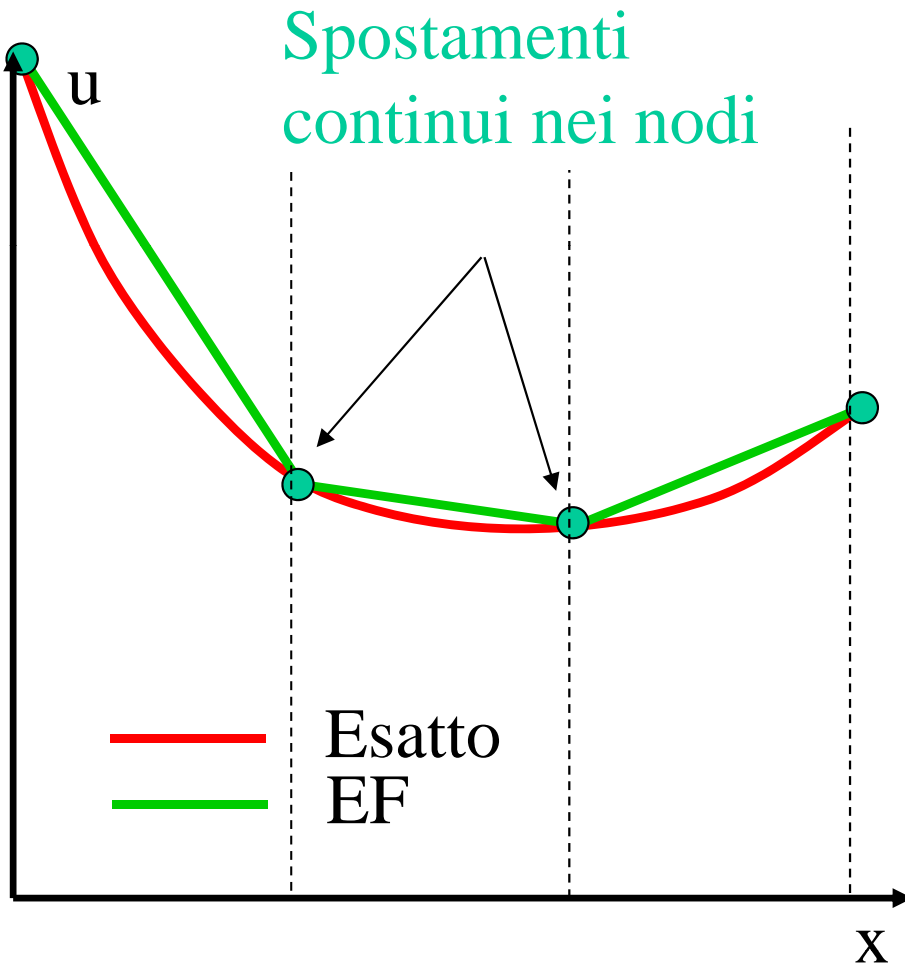
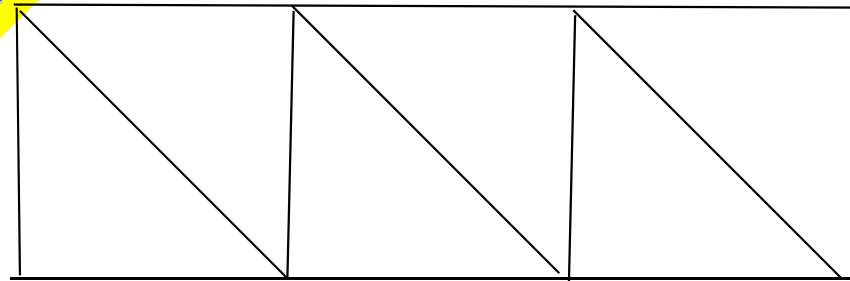


## Andamento effettivo delle tensioni

### Tensioni discontinue nei nodi

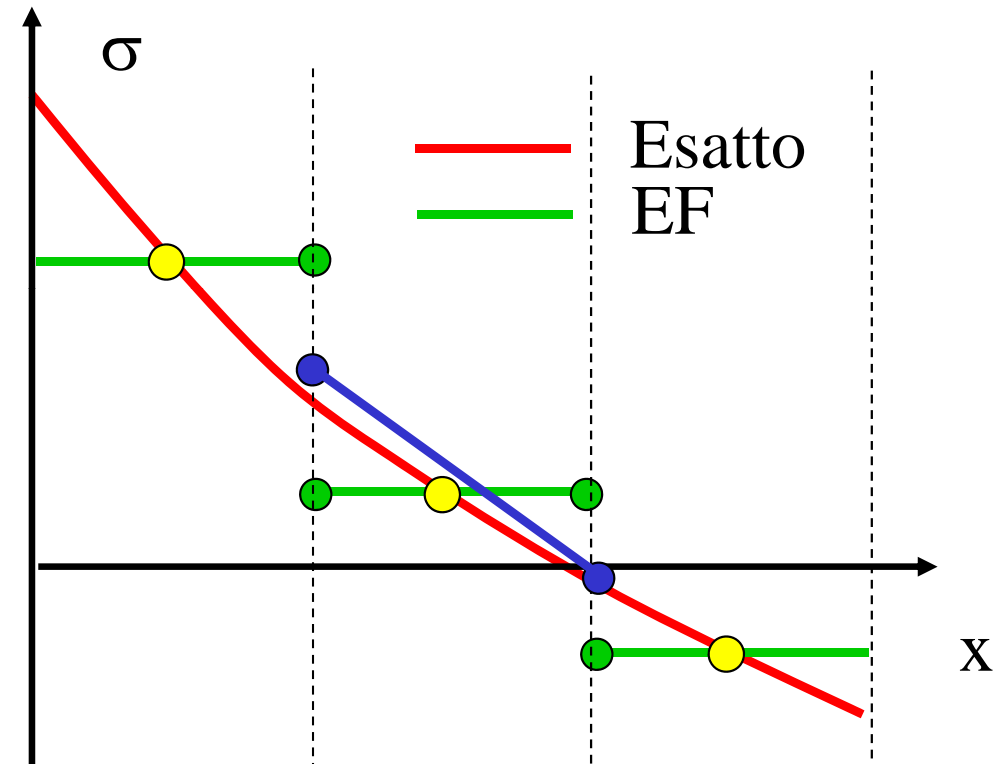


Calcolo di valori mediati nei nodi  
(media aritmetica o altre tecniche)



## Andamento effettivo delle tensioni

### Tensioni discontinue nei nodi

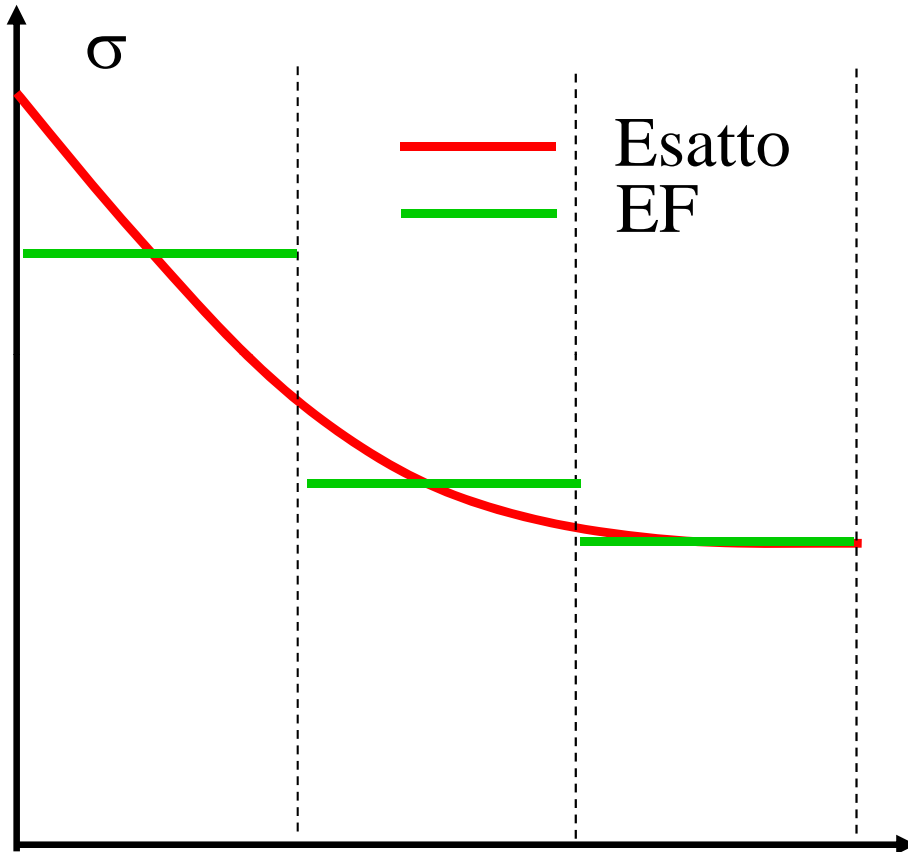


Calcolo di valori mediati nei nodi  
(media aritmetica o altre tecniche)

Interpolazione dei valori mediati nodali  
nelle zone interne (Es. tramite le N)

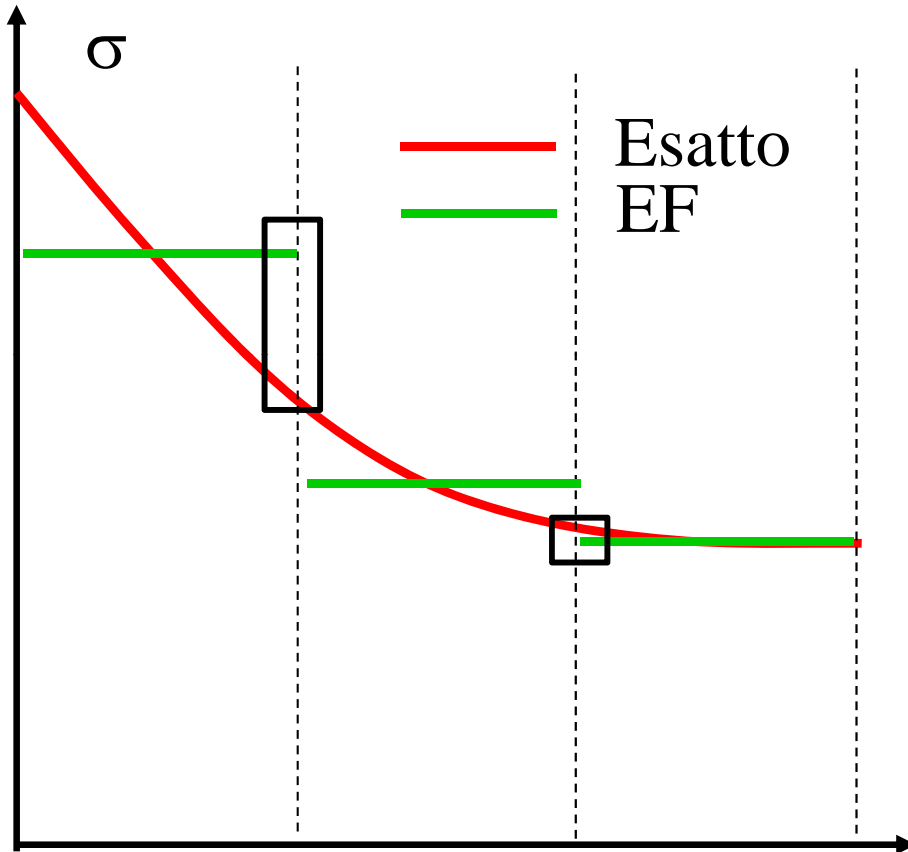


# Dimensioni ottimali degli elementi



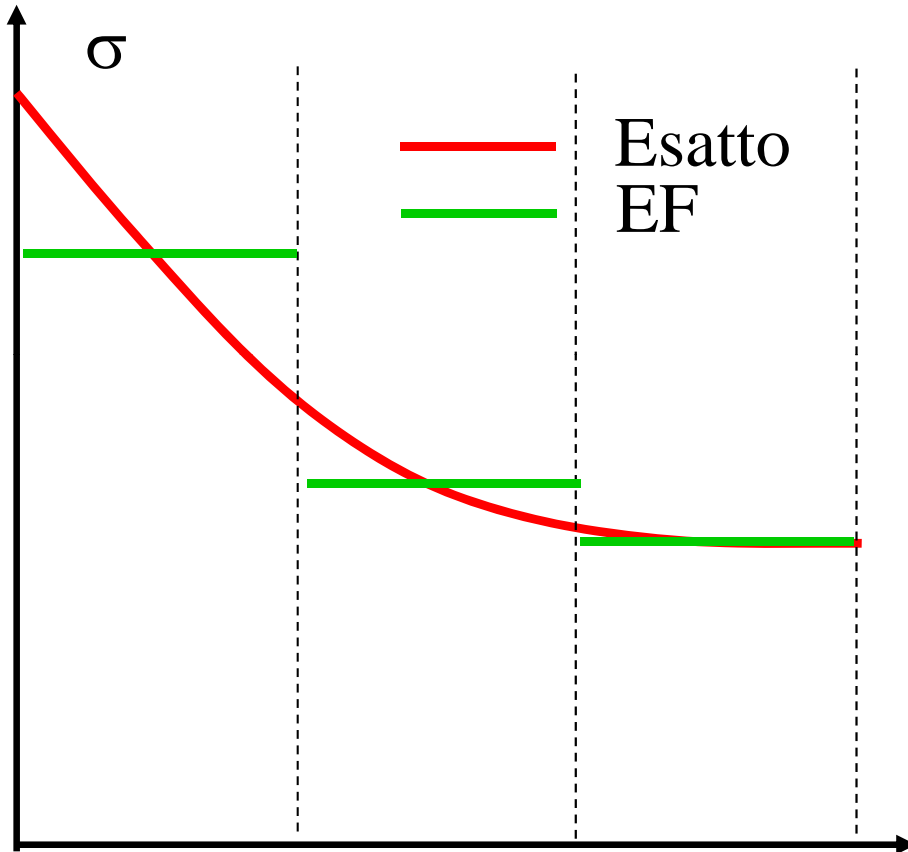


# Dimensioni ottimali degli elementi

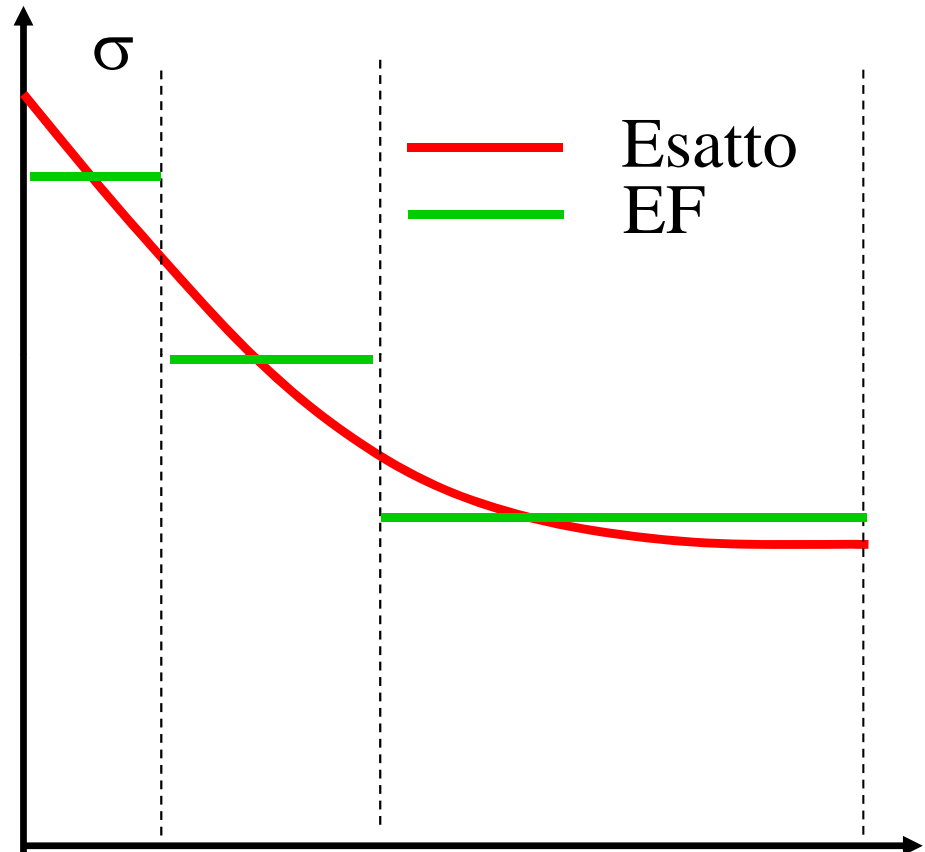




## Dimensioni ottimali degli elementi



Dimensioni elementi  
non ottimali

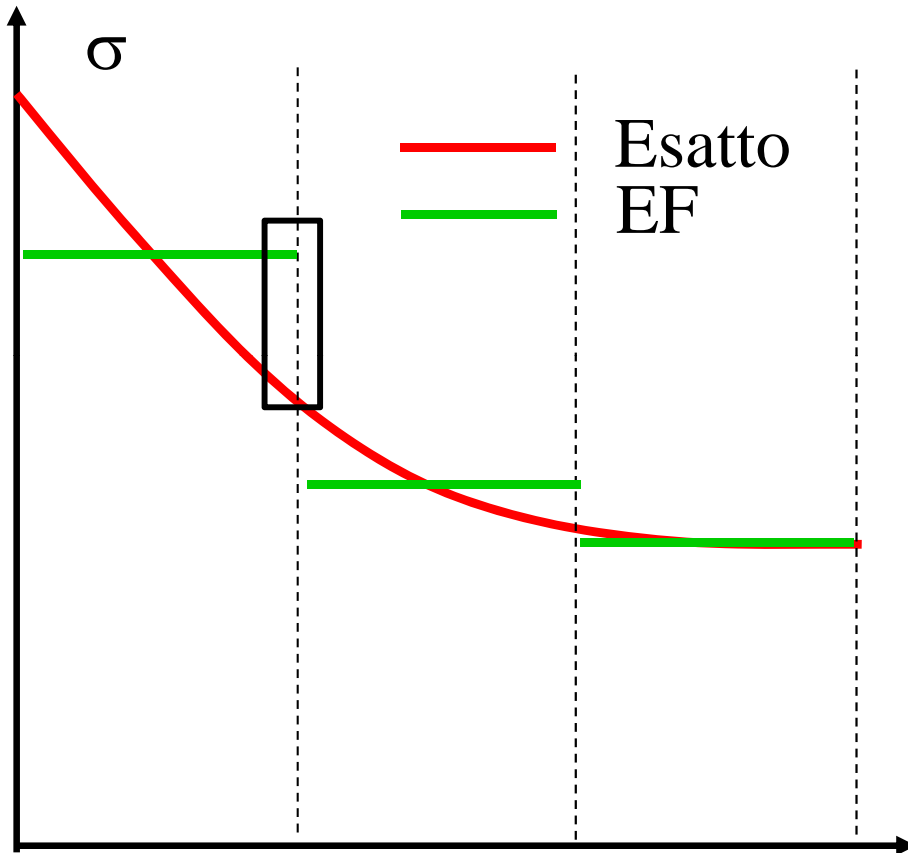


Dimensioni elementi  
ottimali

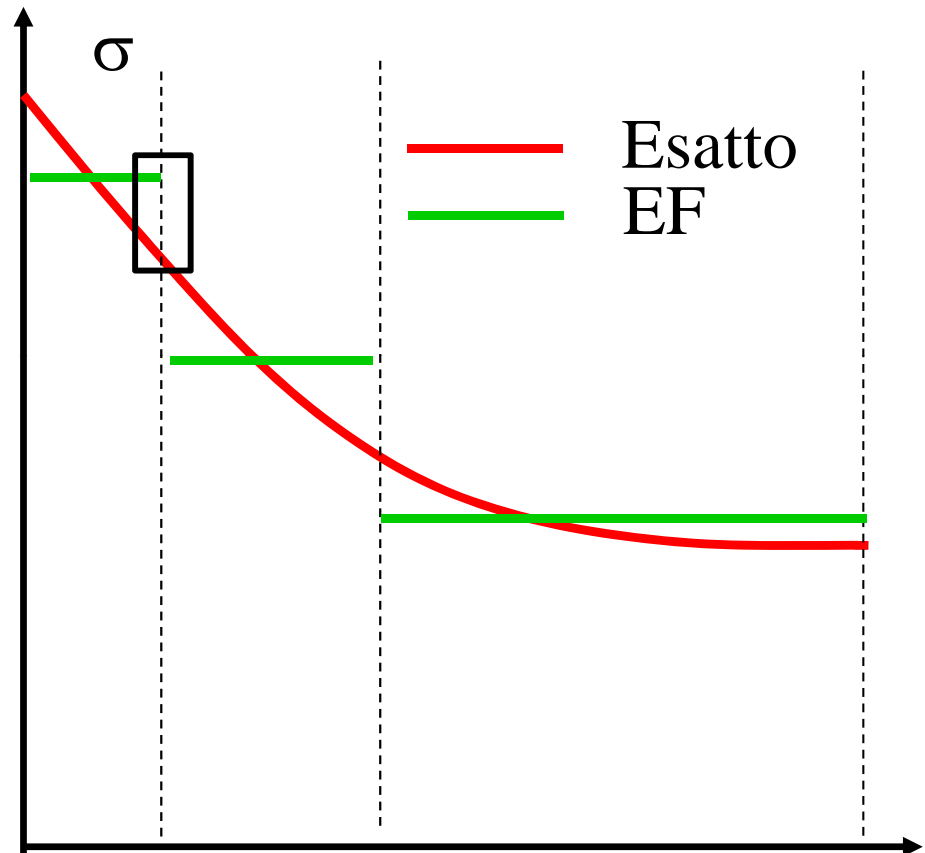




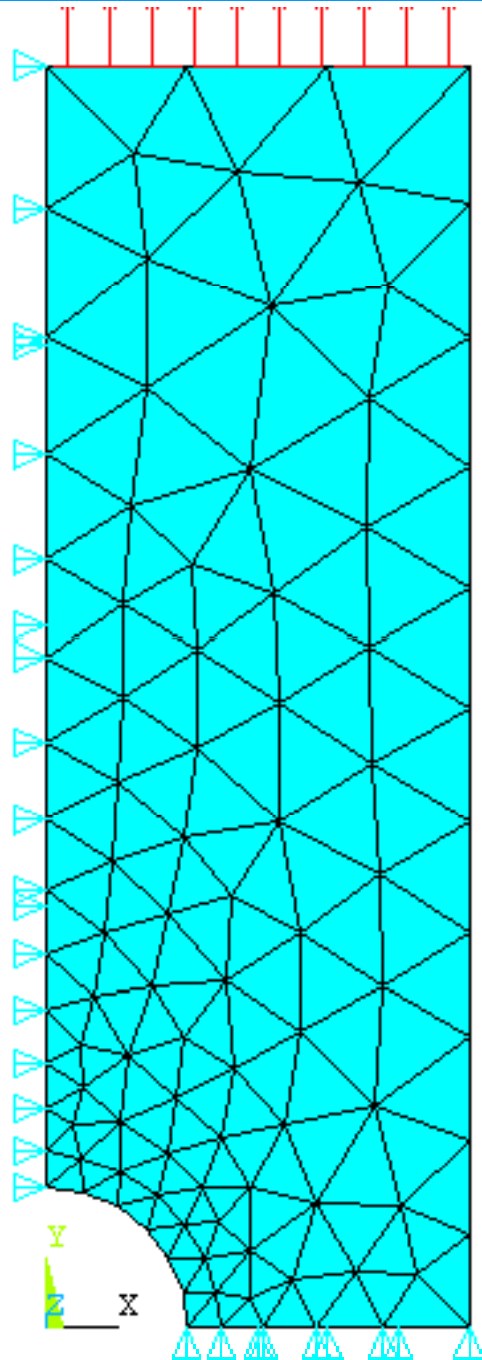
## Dimensioni ottimali degli elementi



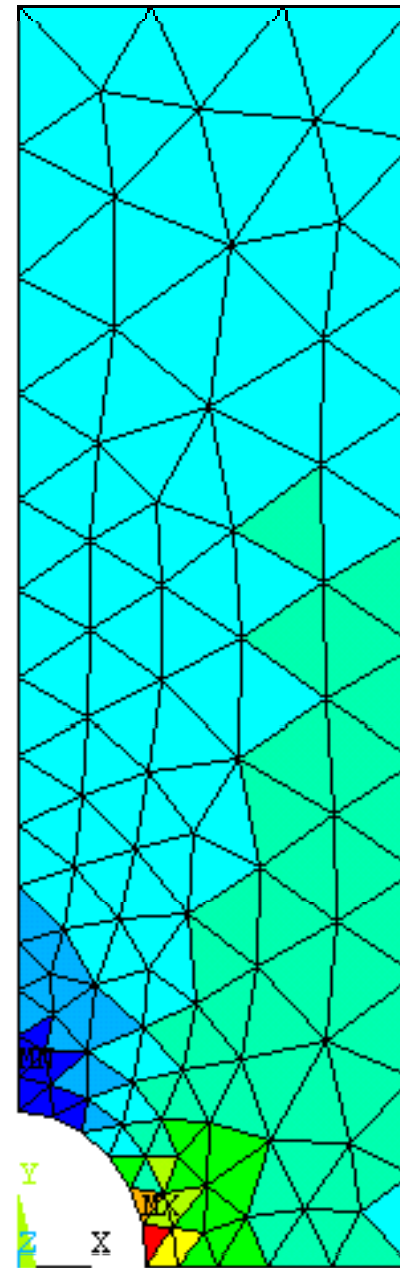
Dimensioni elementi  
non ottimali



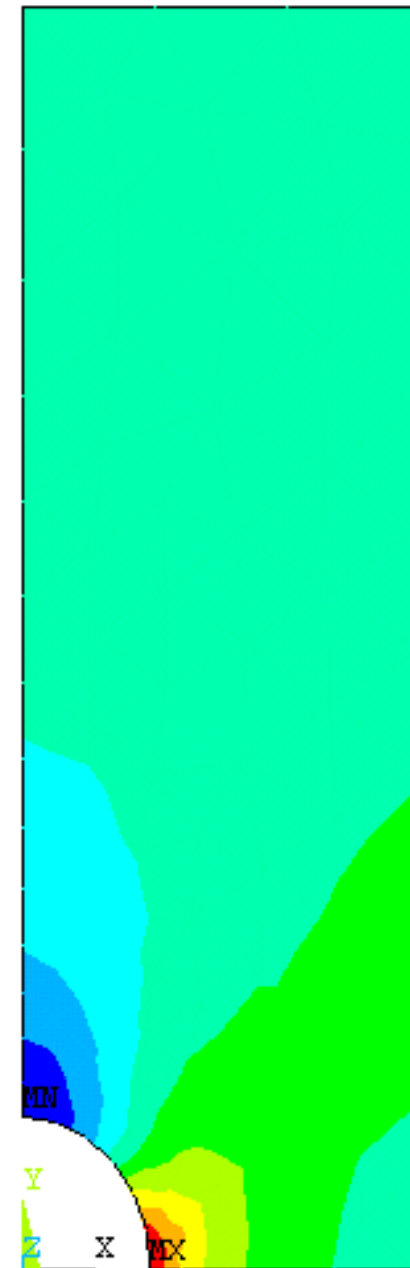
Dimensioni elementi  
ottimali



Modello



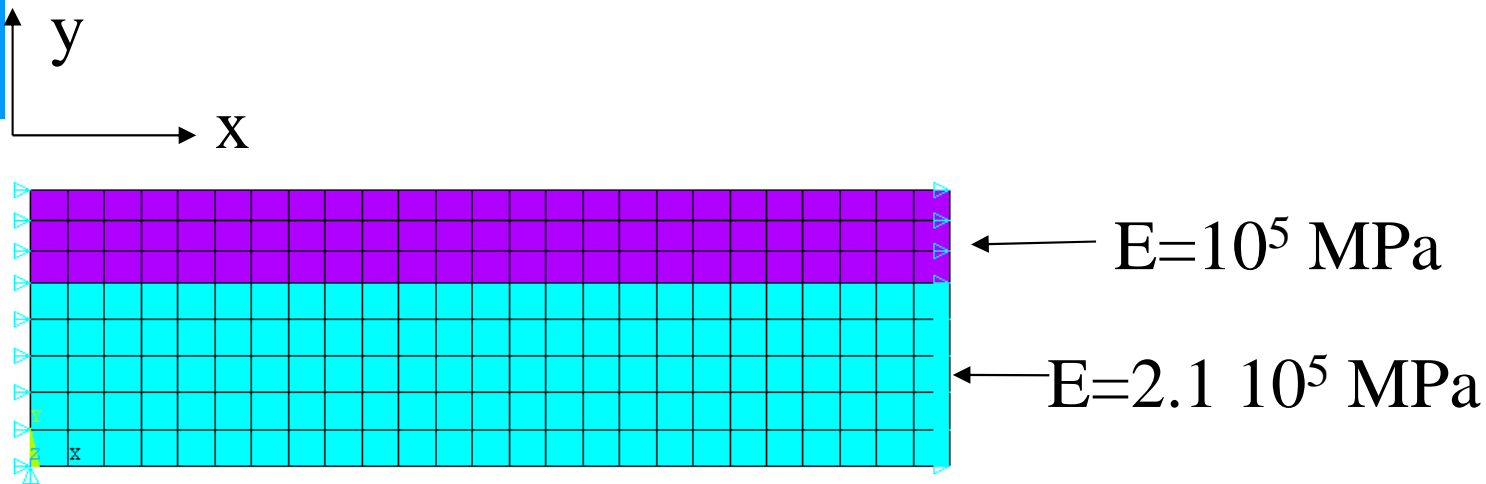
Tensioni  $\sigma_y$  non mediate



Tensioni  $\sigma_y$  mediate

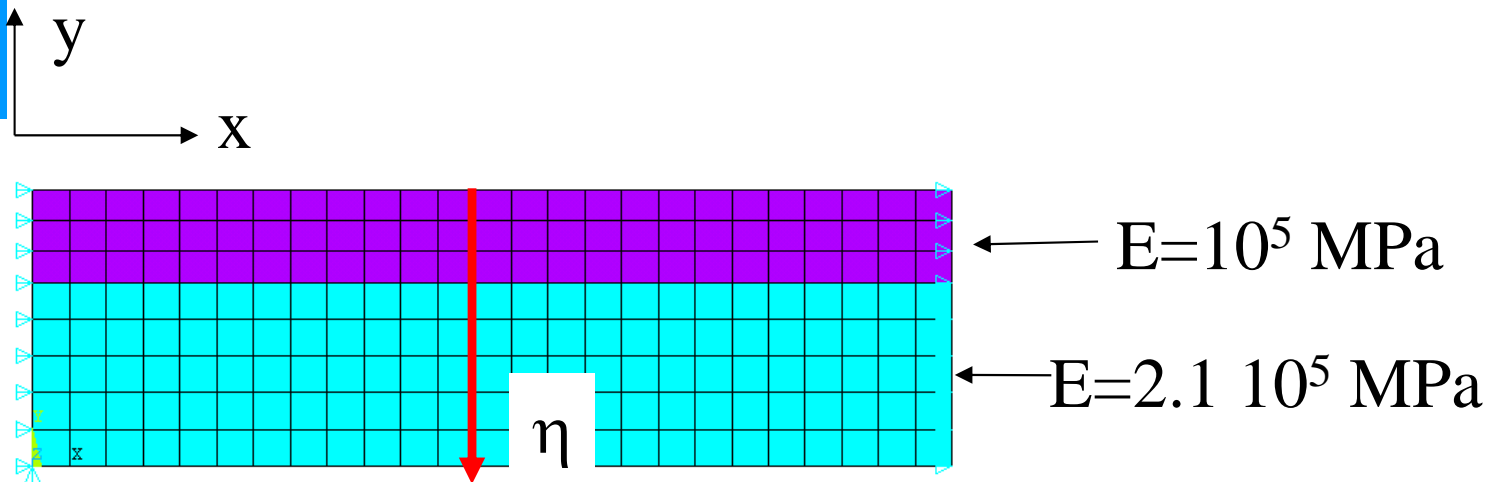


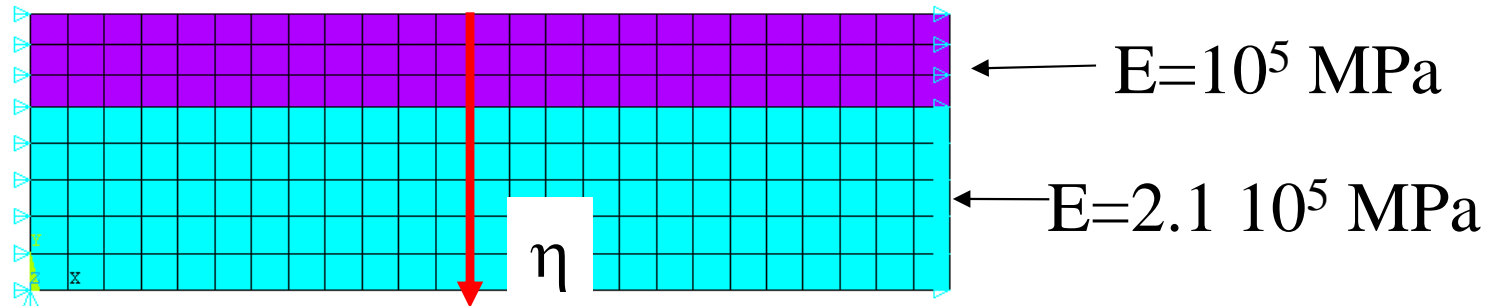
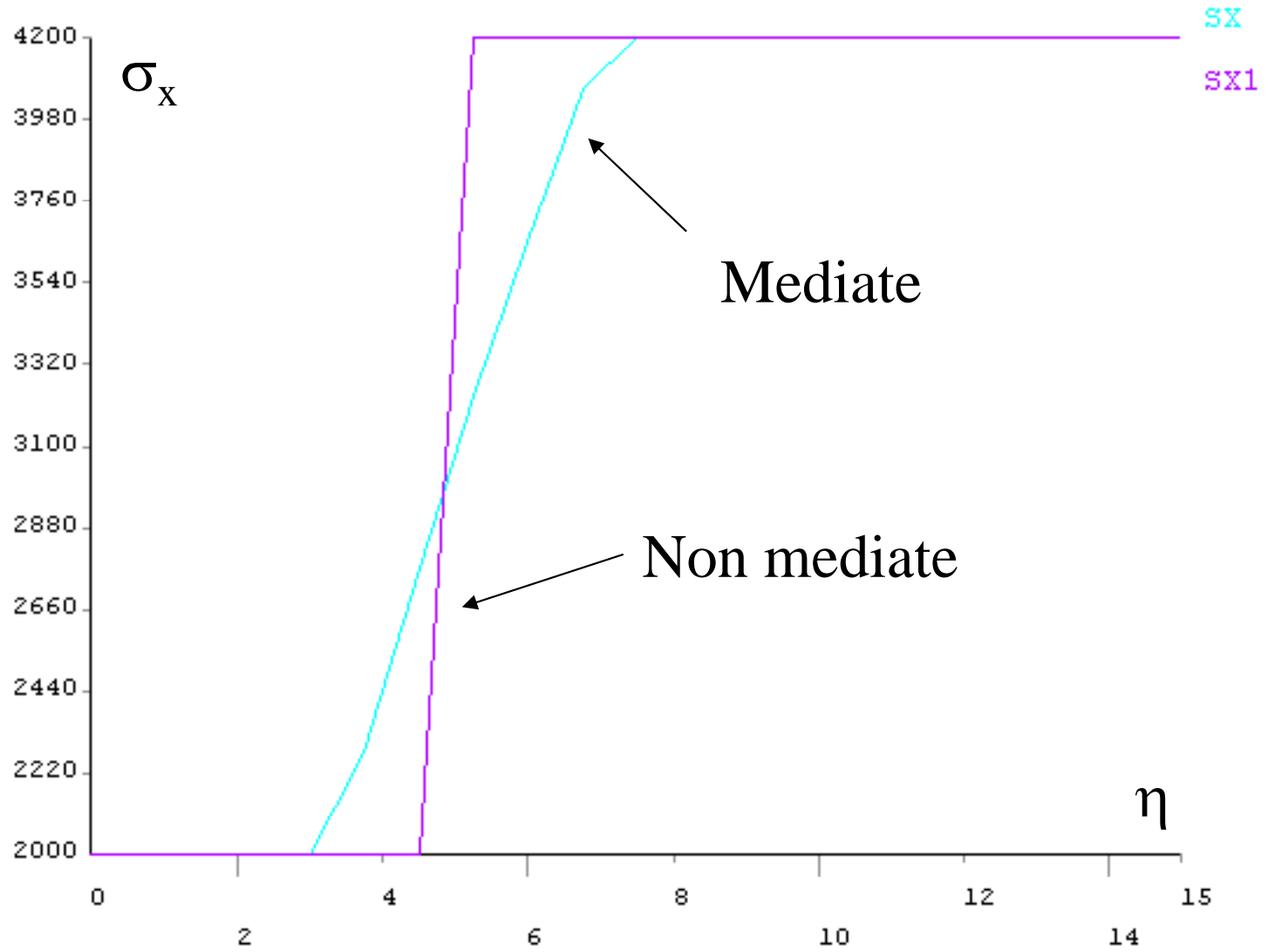
In casi in cui le tensioni sono intrinsecamente discontinue, l'operazione di media nei nodi può diminuire la precisione.  
Esempio 1 : Lastra in due materiali diversi, soggetta ad allungamento uniforme



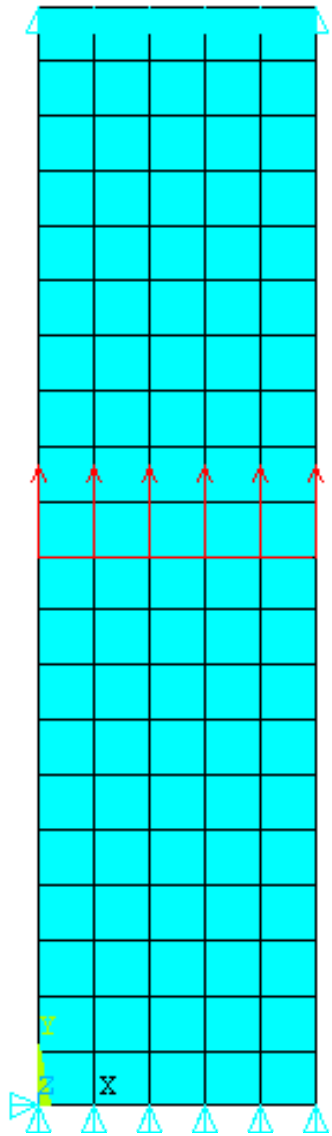


In casi in cui le tensioni sono intrinsecamente discontinue, l'operazione di media nei nodi può diminuire la precisione.  
Esempio 1 : Lastra in due materiali diversi, soggetta ad allungamento uniforme

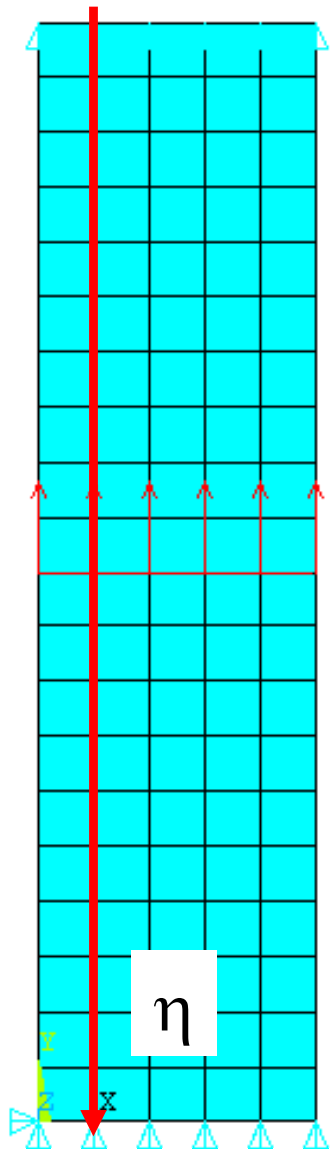




## Esempio 2: lastra incastrata agli estremi e caricata al centro

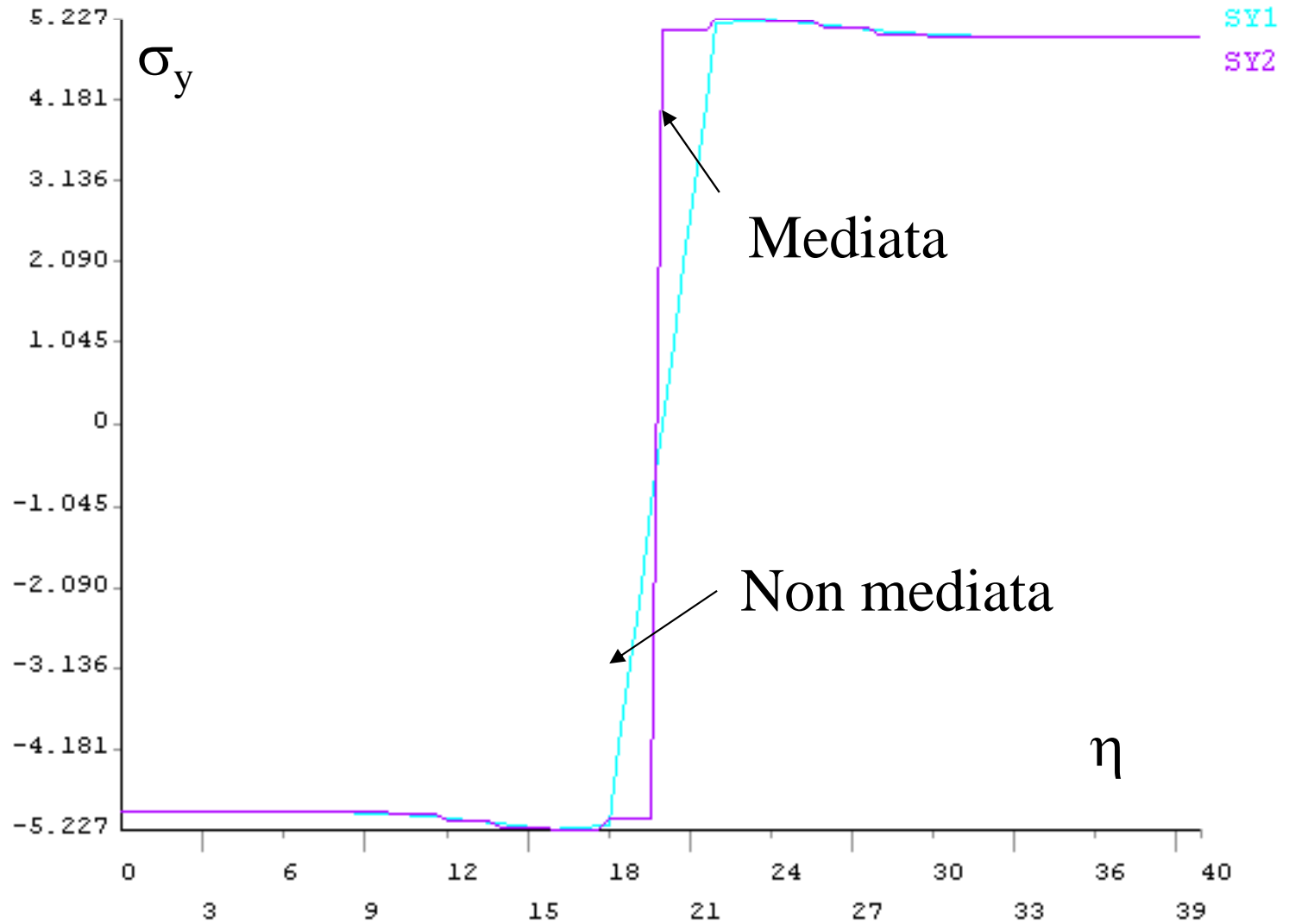
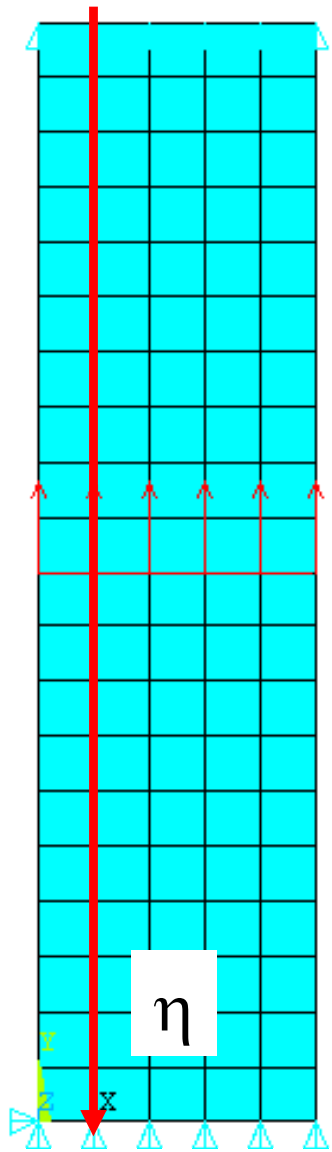


## Esempio 2: lastra incastrata agli estremi e caricata al centro





# Esempio 2: lastra incastrata agli estremi e caricata al centro

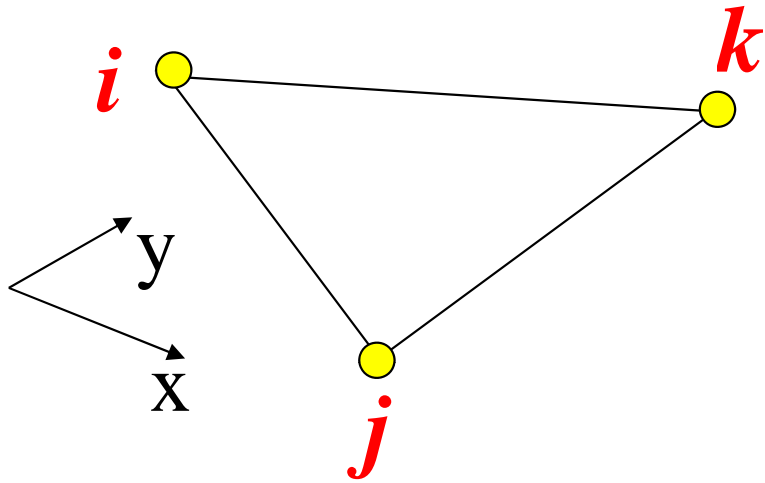


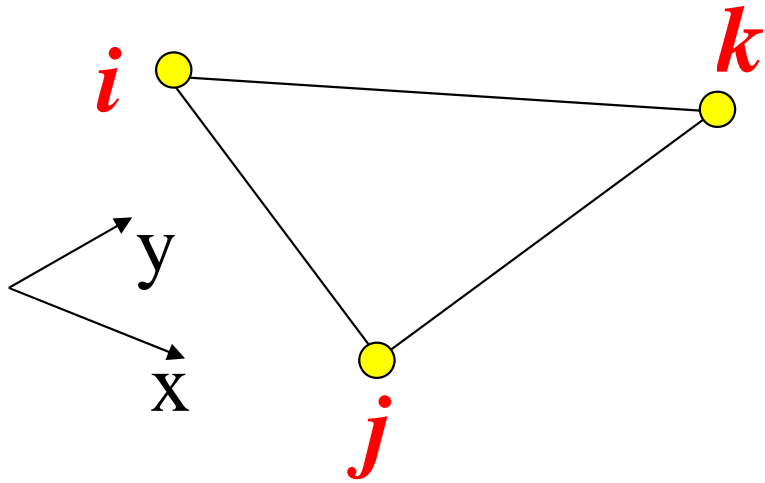




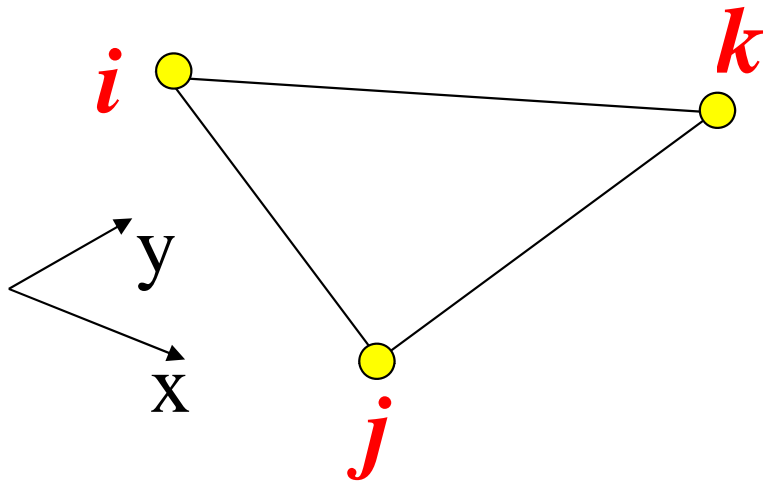
# Aspetti teorici ed applicativi del MEF– Parte I

## Elementi di ordine superiore



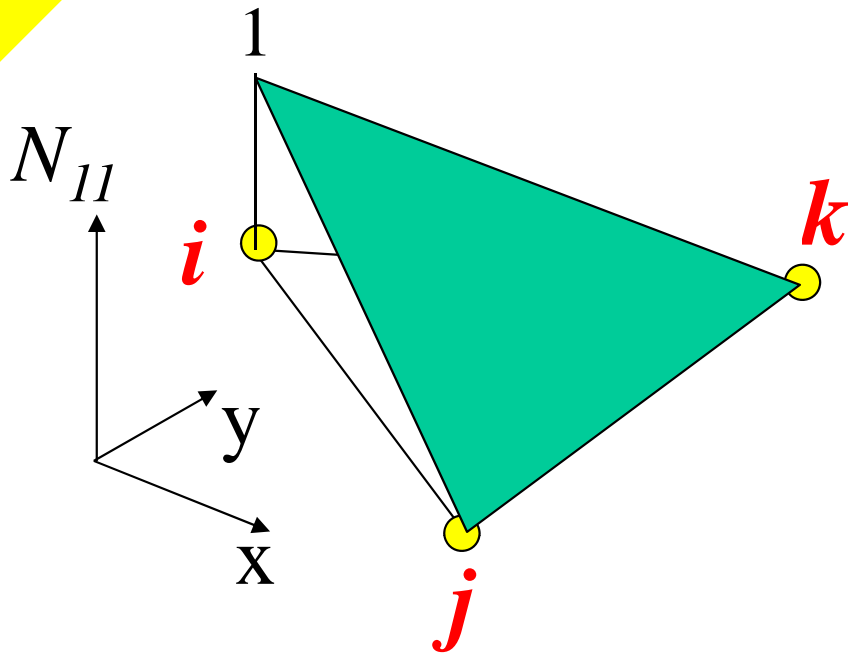


$$\begin{cases} N_{11}(x_i, y_i) = 1 \\ N_{11}(x_j, y_j) = 0 \\ N_{11}(x_k, y_k) = 0 \end{cases}$$



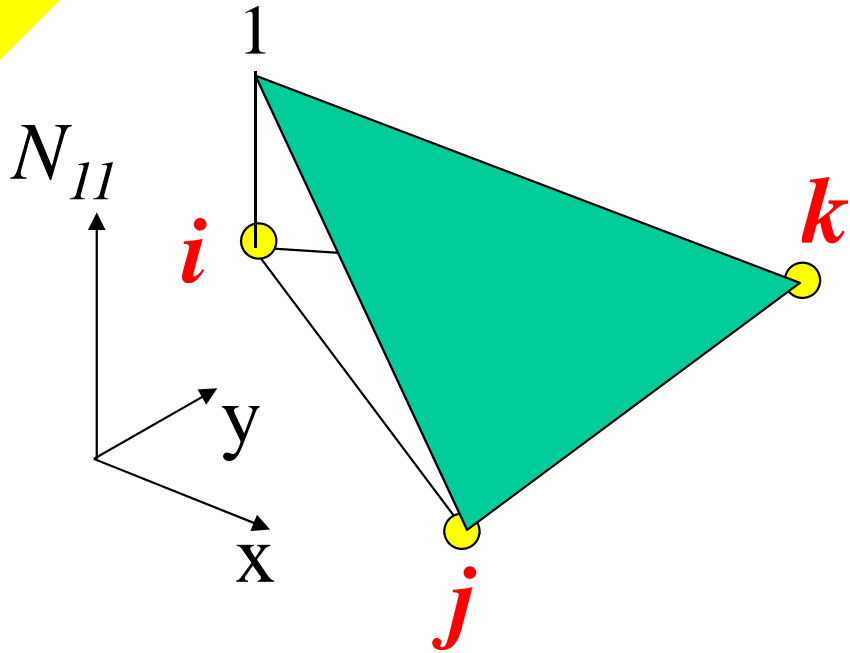
$$\begin{cases} N_{11}(x_i, y_i) = 1 \\ N_{11}(x_j, y_j) = 0 \\ N_{11}(x_k, y_k) = 0 \end{cases}$$

$$N_{lm}^e(x, y) = A_{lm} + B_{lm} \cdot x + C_{lm} \cdot y$$

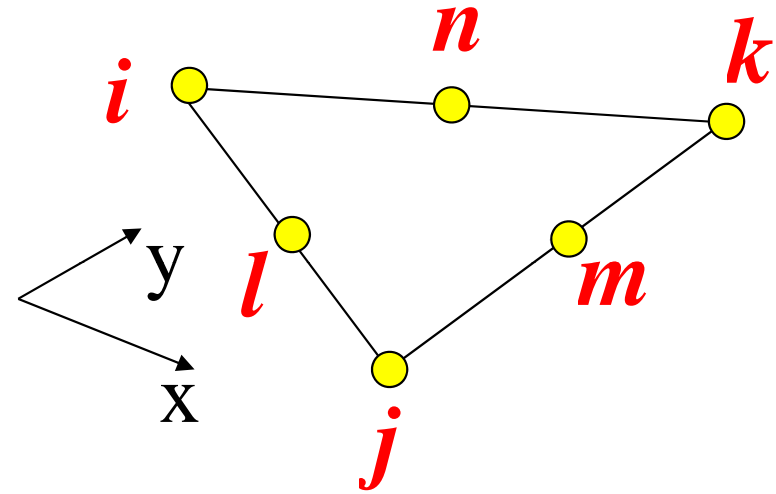


$$\begin{cases} N_{11}(x_i, y_i) = 1 \\ N_{11}(x_j, y_j) = 0 \\ N_{11}(x_k, y_k) = 0 \end{cases}$$

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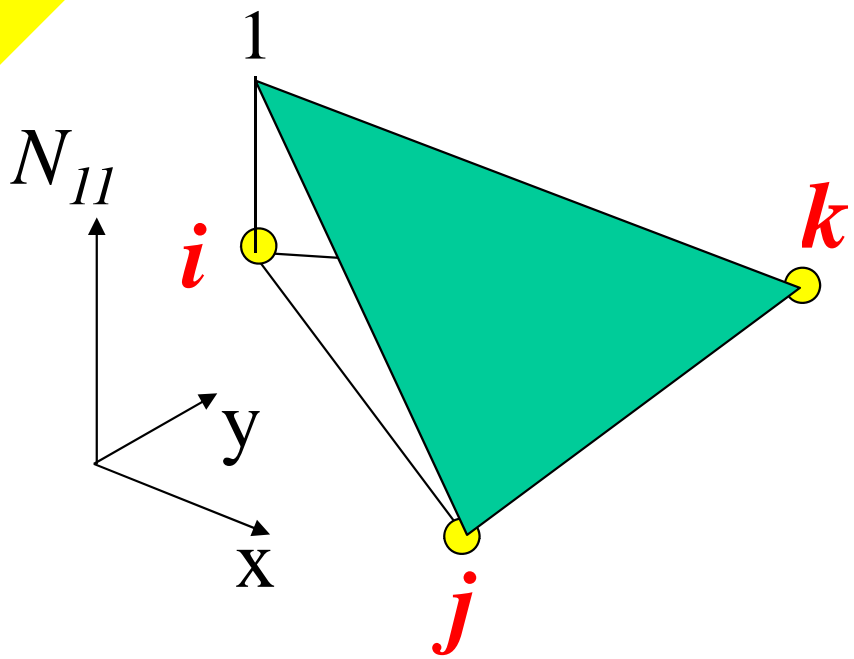


$$N_{lm}^e(x, y) = A_{lm} + B_{lm} \cdot x + C_{lm} \cdot y$$

# Elementi di ordine superiore

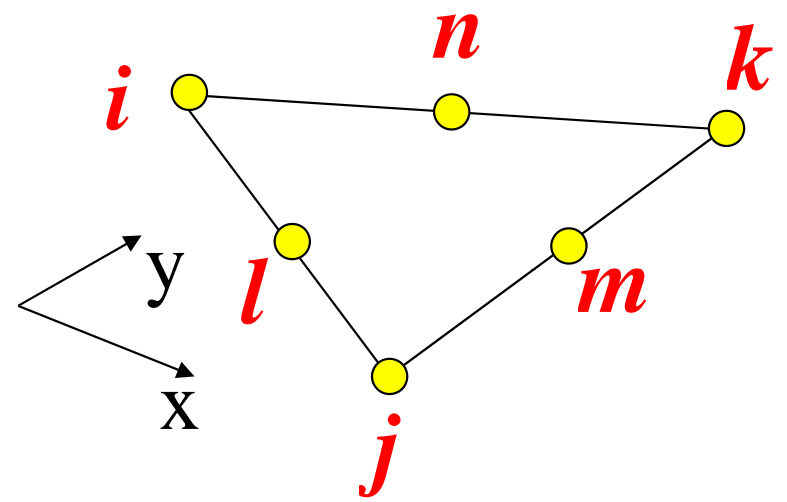


Scuola di Dottorato in Ingegneria "Leonardo da Vinci"



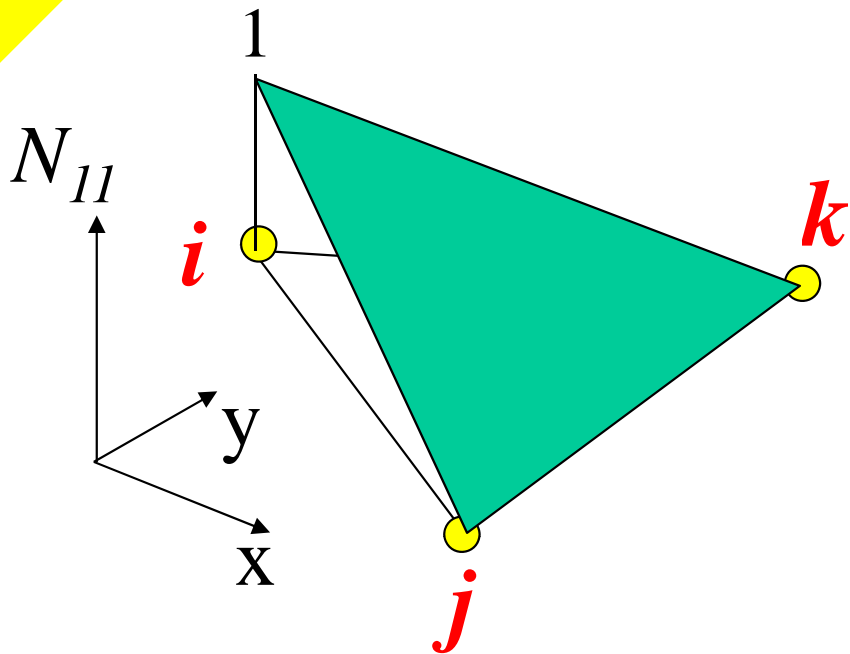
$$\begin{cases} N_{11}(x_i, y_i) = 1 \\ N_{11}(x_j, y_j) = 0 \\ N_{11}(x_k, y_k) = 0 \end{cases}$$

$$N_{lm}^e(x, y) = A_{lm} + B_{lm} \cdot x + C_{lm} \cdot y$$



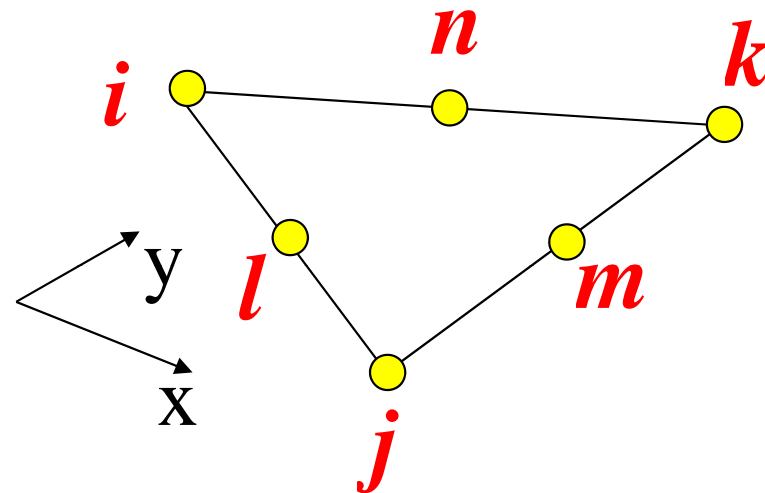
$$\begin{cases} N_{11}(x_i, y_i) = 1 \\ N_{11}(x_j, y_j) = 0 \\ N_{11}(x_k, y_k) = 0 \end{cases} \quad \begin{cases} N_{11}(x_l, y_l) = 0 \\ N_{11}(x_m, y_m) = 0 \\ N_{11}(x_n, y_n) = 0 \end{cases}$$

## Elementi di ordine superiore



$$\begin{cases} N_{11}(x_i, y_i) = 1 \\ N_{11}(x_j, y_j) = 0 \\ N_{11}(x_k, y_k) = 0 \end{cases}$$

$$N_{lm}^e(x, y) = A_{lm} + B_{lm} \cdot x + C_{lm} \cdot y$$

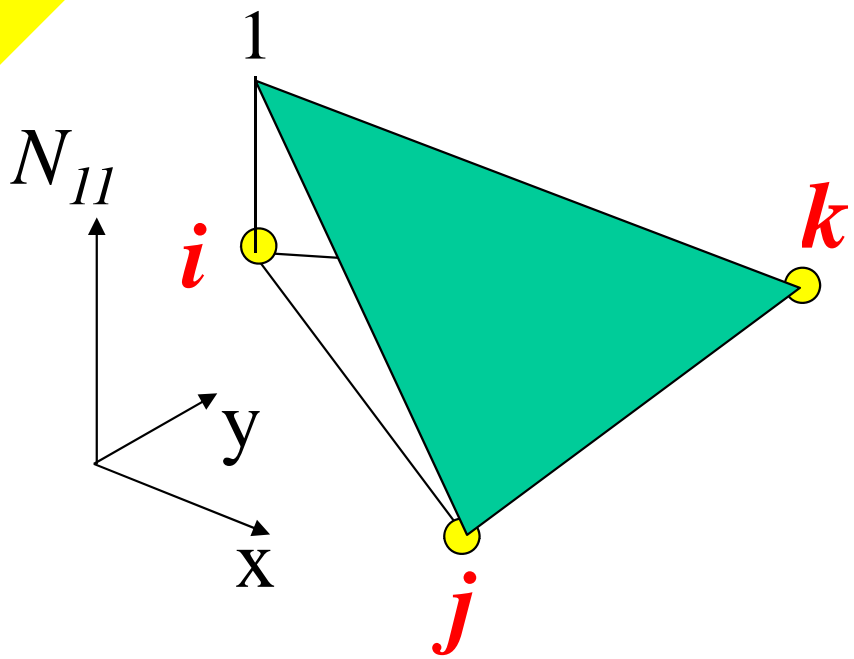


$$\begin{cases} N_{11}(x_i, y_i) = 1 \\ N_{11}(x_j, y_j) = 0 \\ N_{11}(x_k, y_k) = 0 \end{cases} \quad \begin{cases} N_{11}(x_l, y_l) = 0 \\ N_{11}(x_m, y_m) = 0 \\ N_{11}(x_n, y_n) = 0 \end{cases}$$

$$N_{lm}^e(x, y) = A_{lm} + B_{lm} \cdot x + C_{lm} \cdot y + D_{lm} \cdot x^2 + E_{lm} \cdot y^2 + F_{lm} \cdot xy$$

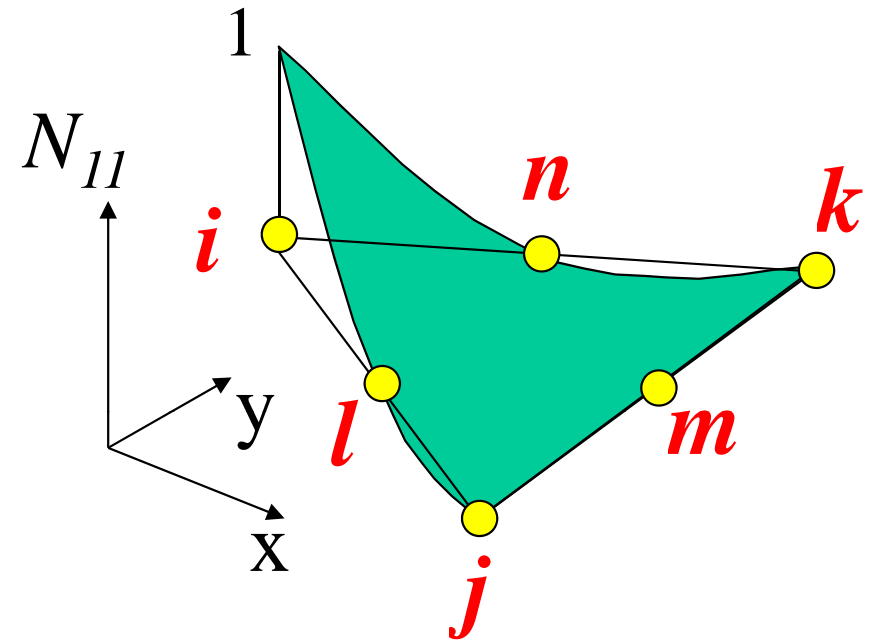
Elementi di ordine superiore

Scuola di Dottorato in Ingegneria "Leonardo da Vinci"



$$\begin{cases} N_{11}(x_i, y_i) = 1 \\ N_{11}(x_j, y_j) = 0 \\ N_{11}(x_k, y_k) = 0 \end{cases}$$

$$N_{lm}^e(x, y) = A_{lm} + B_{lm} \cdot x + C_{lm} \cdot y$$



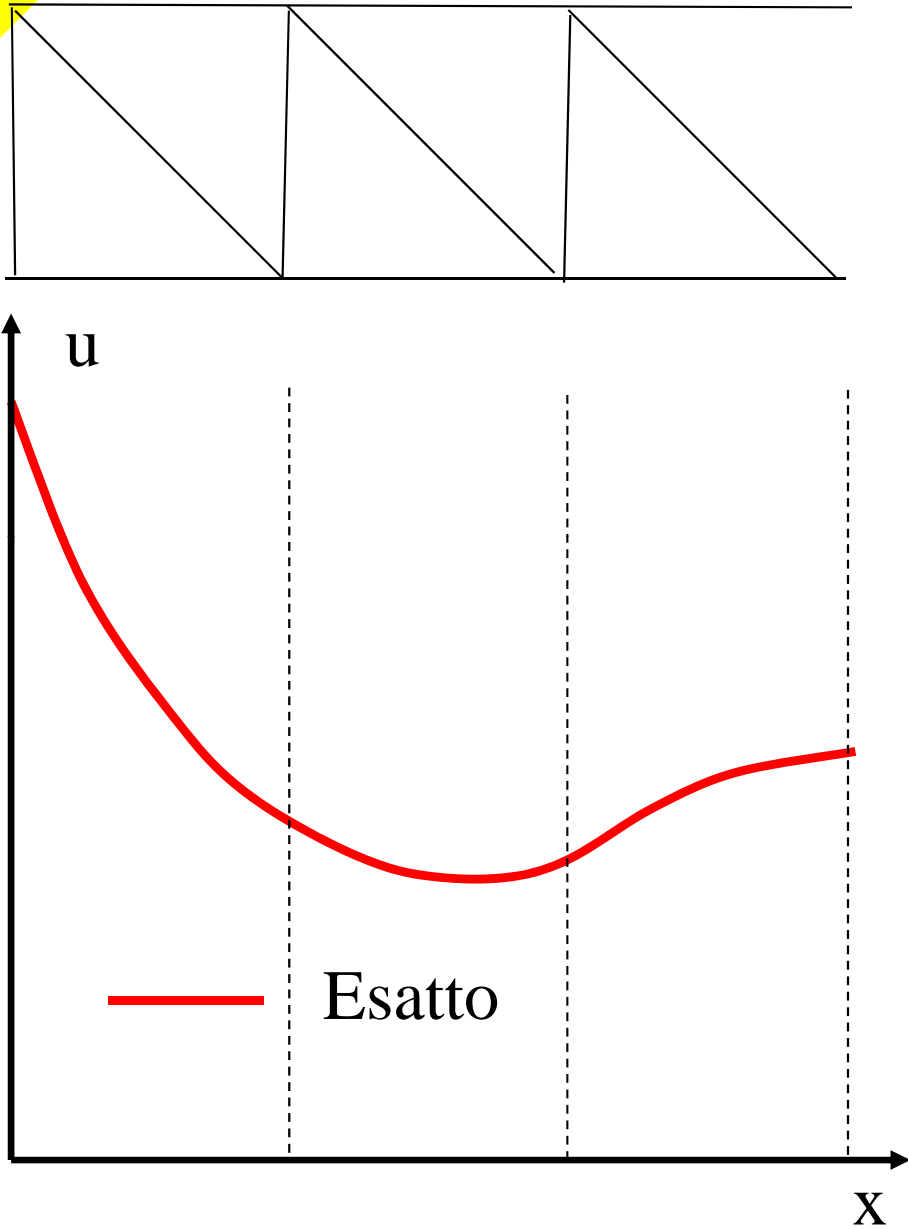
$$\begin{cases} N_{11}(x_i, y_i) = 1 \\ N_{11}(x_j, y_j) = 0 \\ N_{11}(x_k, y_k) = 0 \end{cases} \quad \begin{cases} N_{11}(x_l, y_l) = 0 \\ N_{11}(x_m, y_m) = 0 \\ N_{11}(x_n, y_n) = 0 \end{cases}$$

$$N_{lm}^e(x, y) = A_{lm} + B_{lm} \cdot x + C_{lm} \cdot y + D_{lm} \cdot x^2 + E_{lm} \cdot y^2 + F_{lm} \cdot xy$$



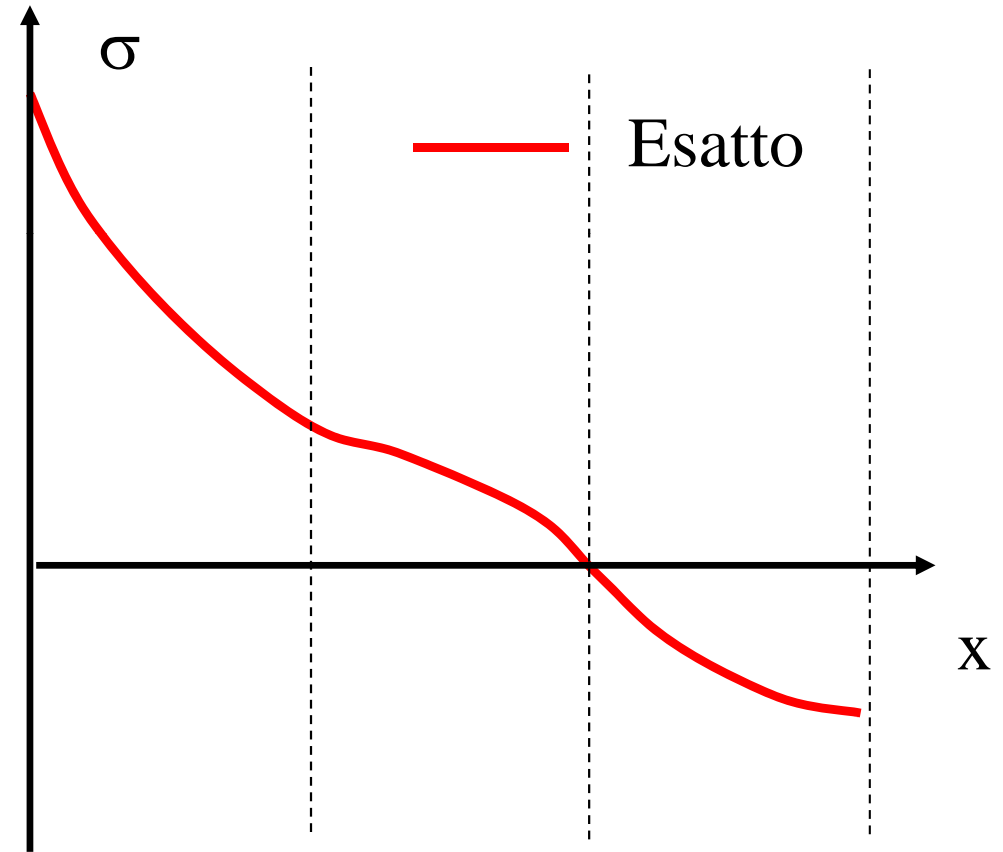
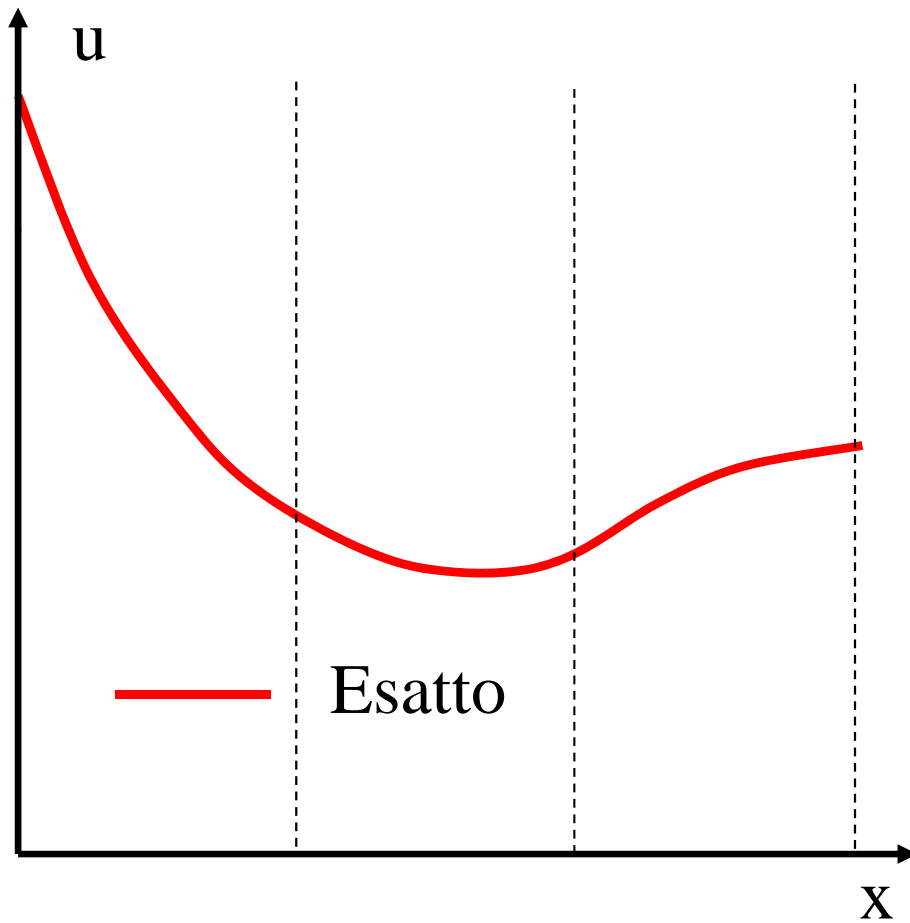
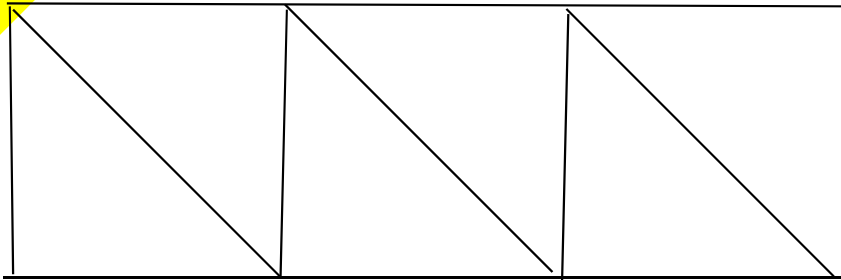


# Elemento con F.ne Forma quadratica



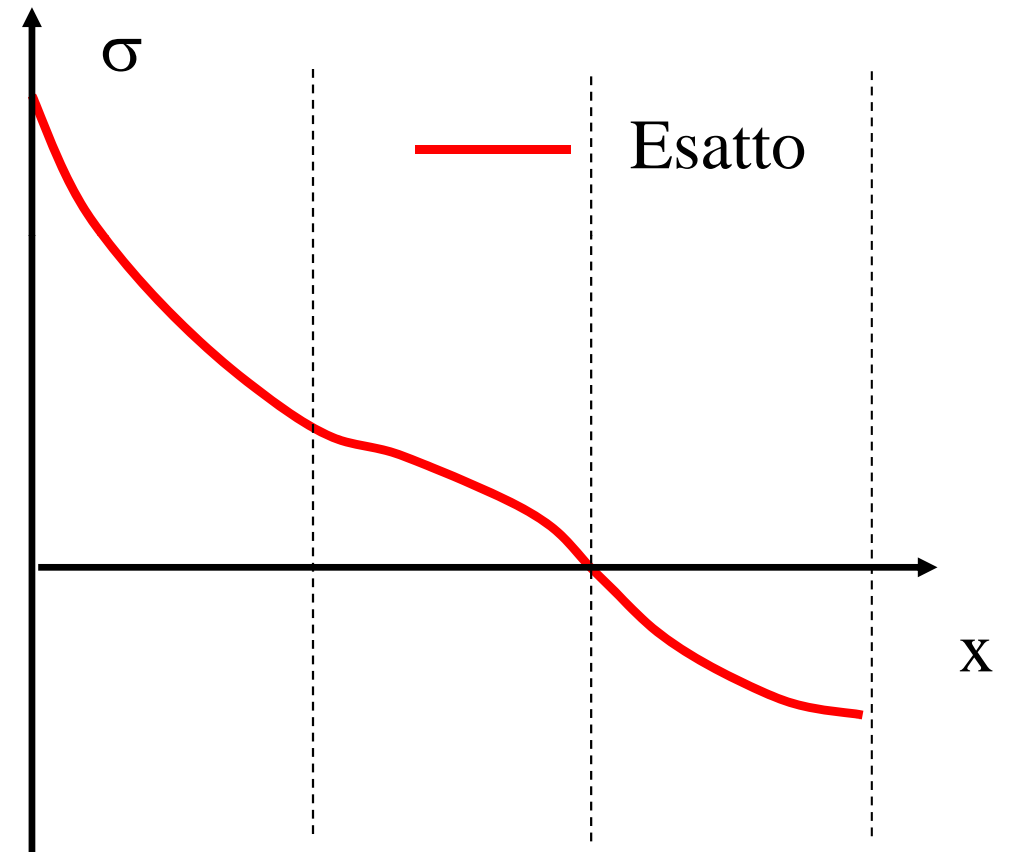
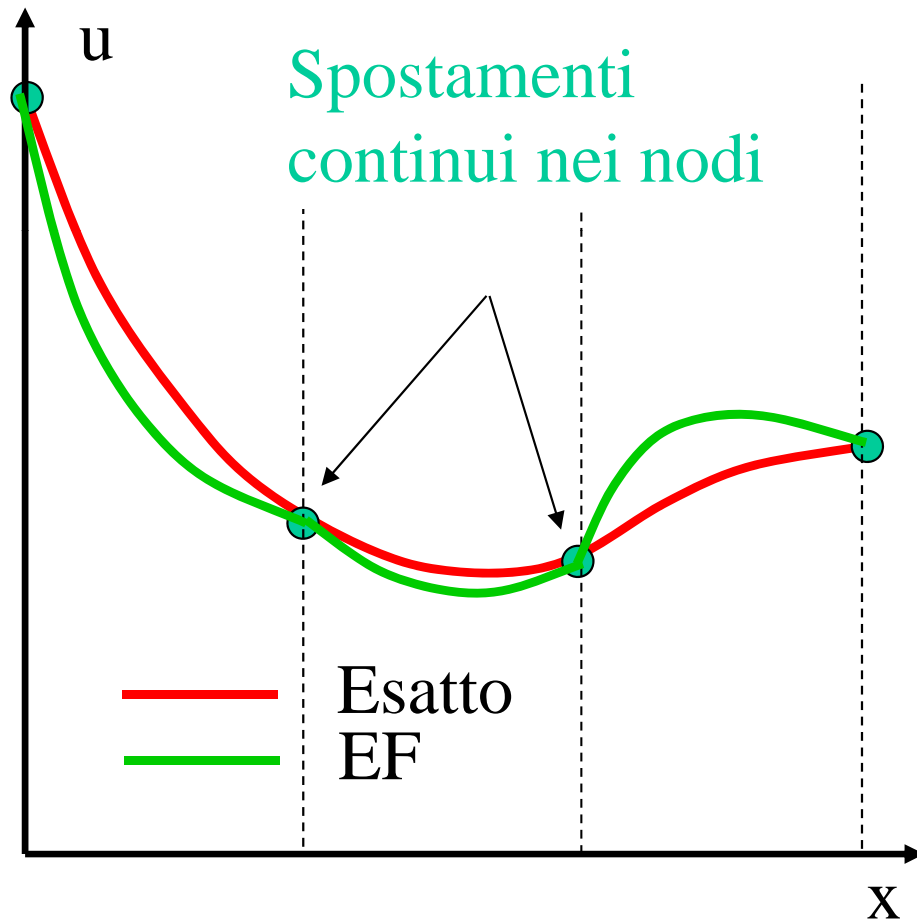
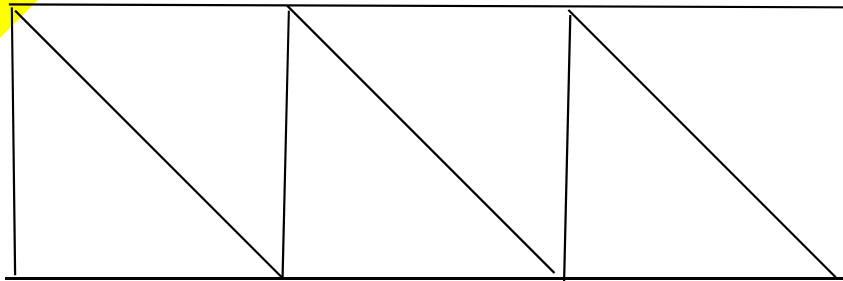


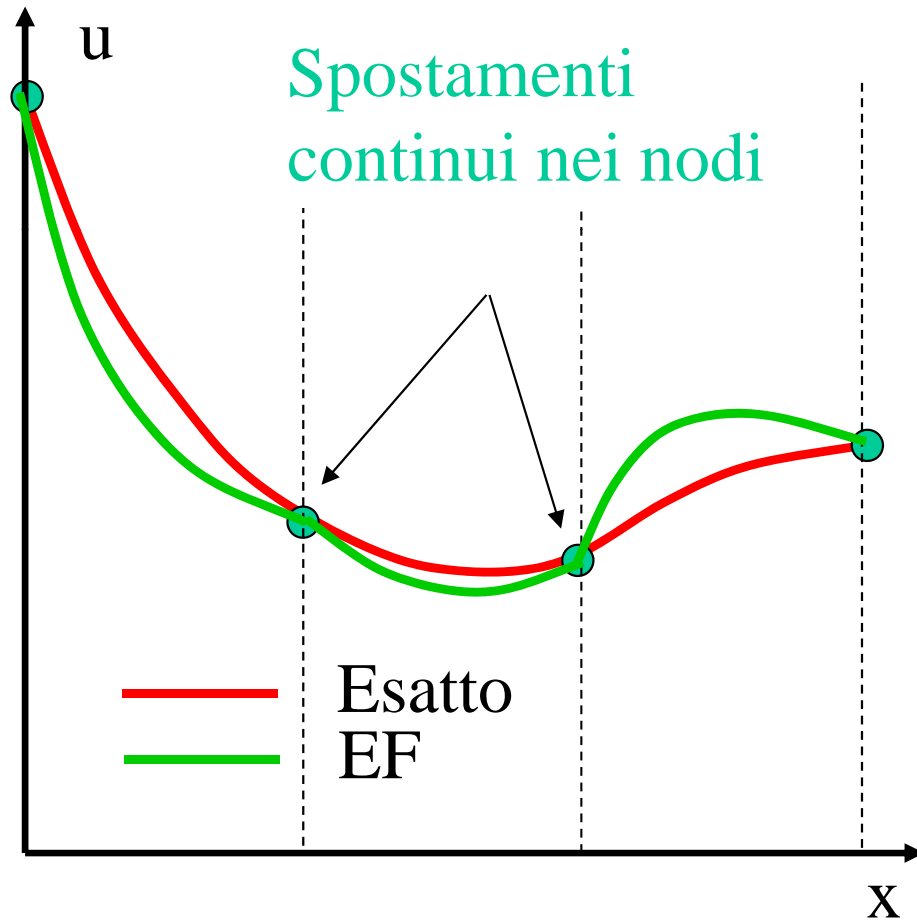
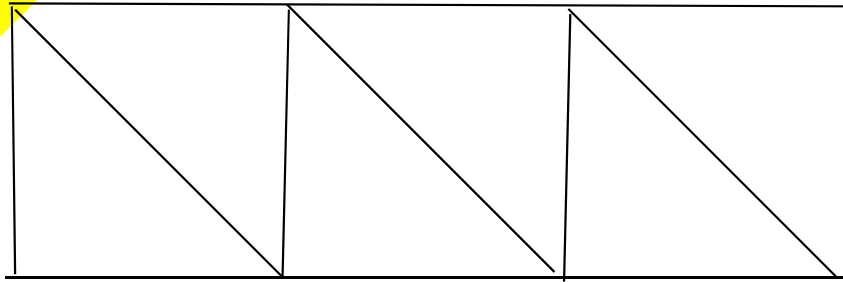
# Elemento con F.ne Forma quadratica





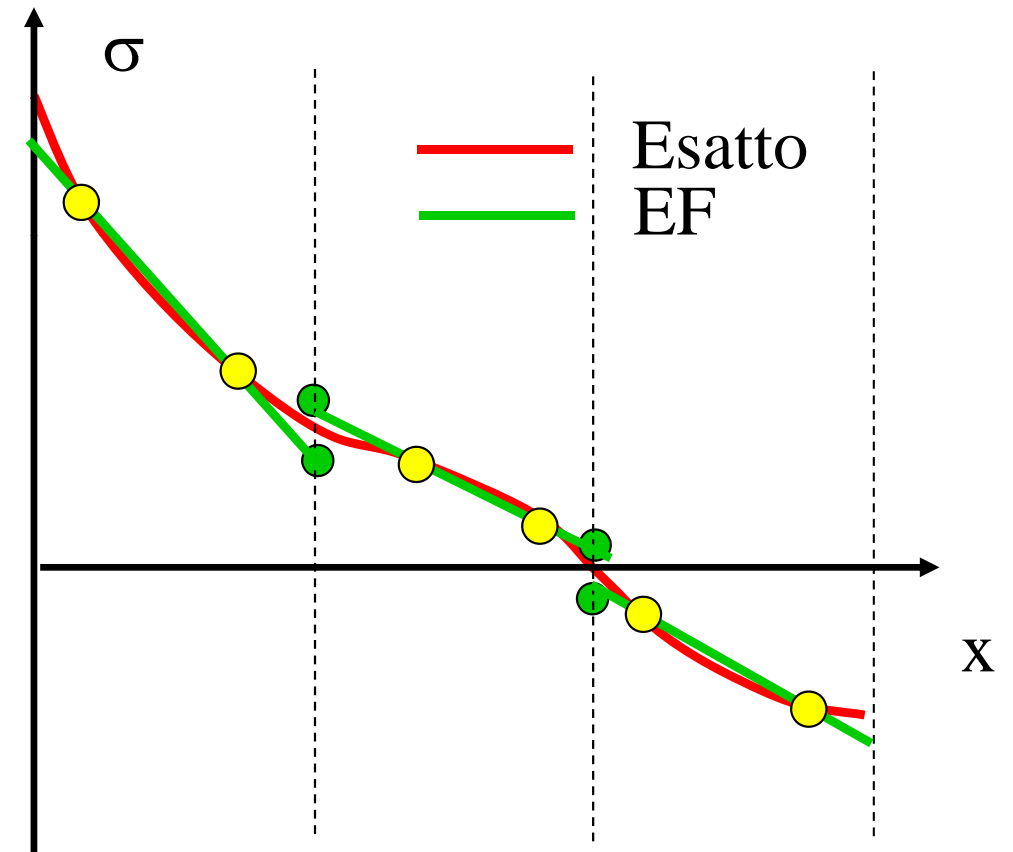
# Elemento con F.ne Forma quadratica





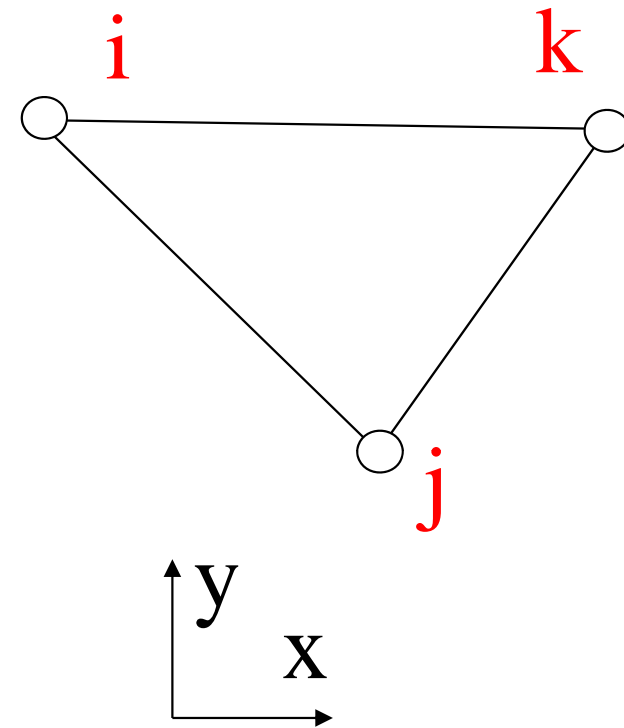
## Elemento con F.ne Forma quadratica

Tensioni discontinue nei nodi



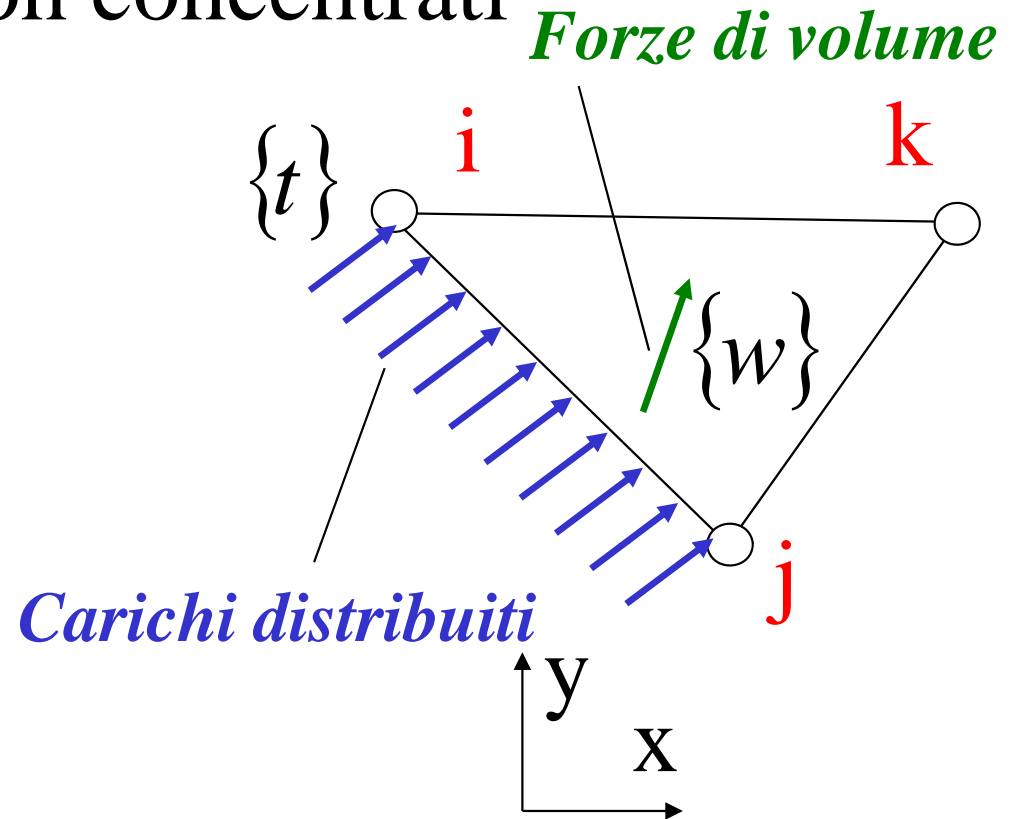


# Carichi non concentrati



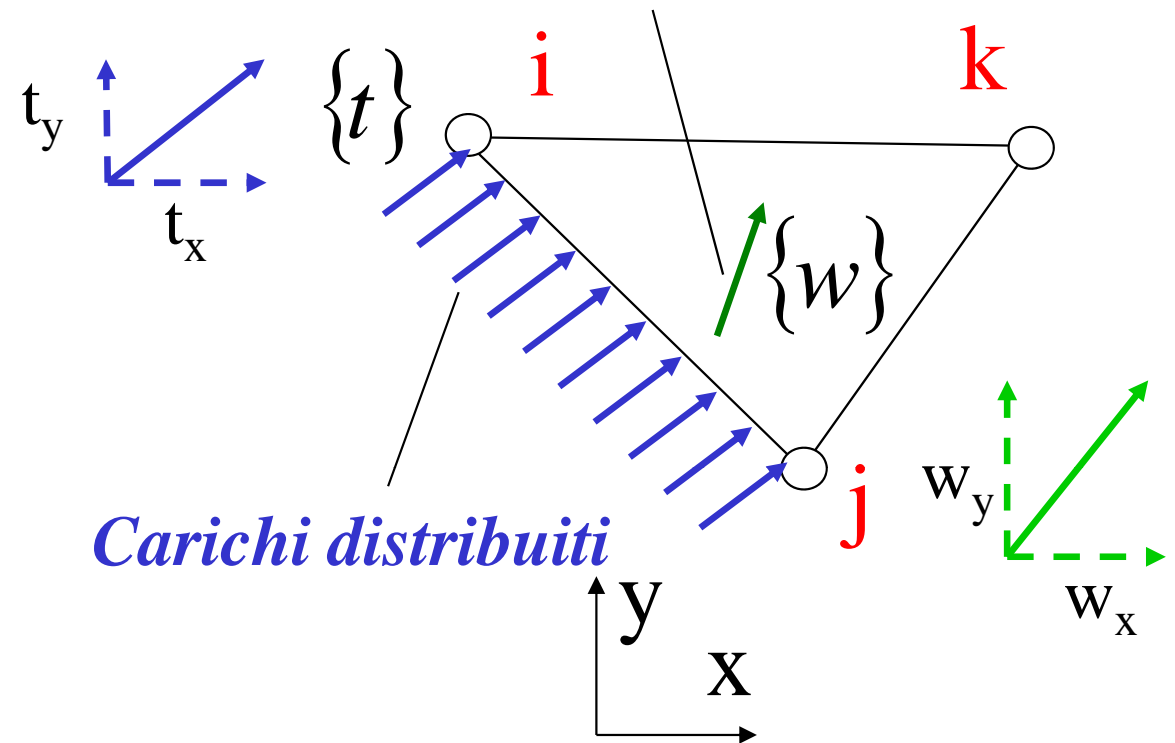


# Carichi non concentrati



# Carichi non concentrati

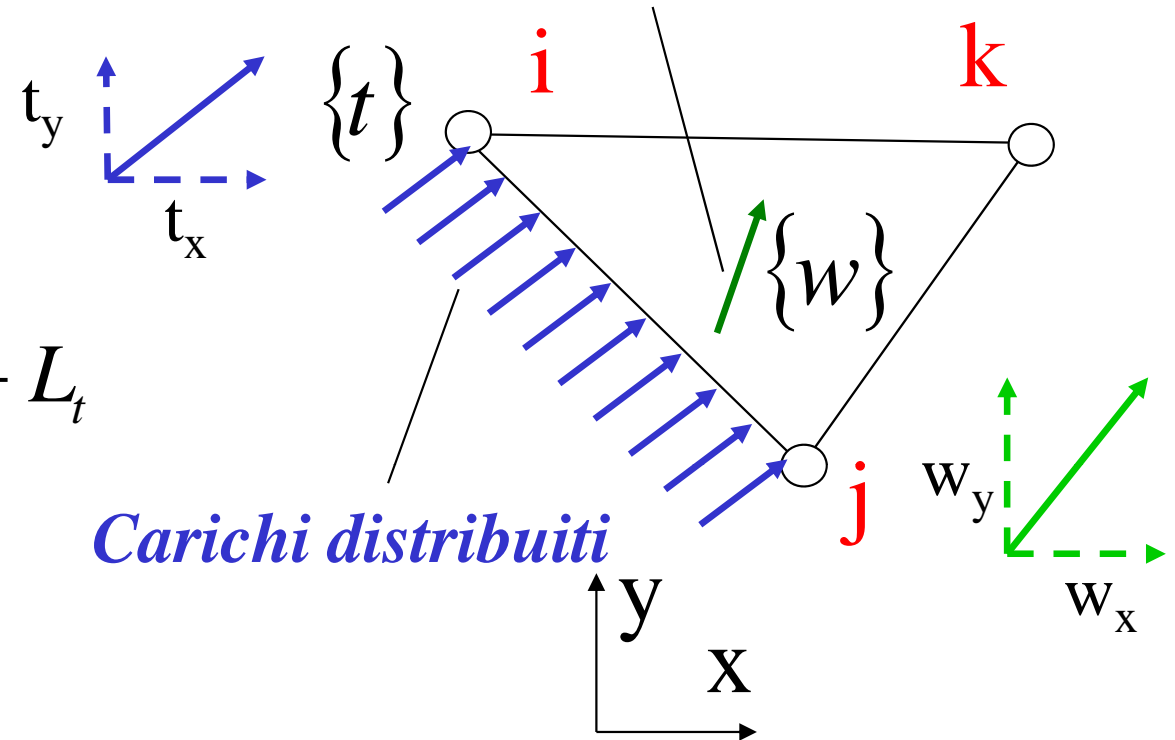
*Forze di volume*



# Carichi non concentrati

*Forze di volume*

$$L_{est} = \{\delta U^e\}^T \{P^e\} + L_W + L_t$$







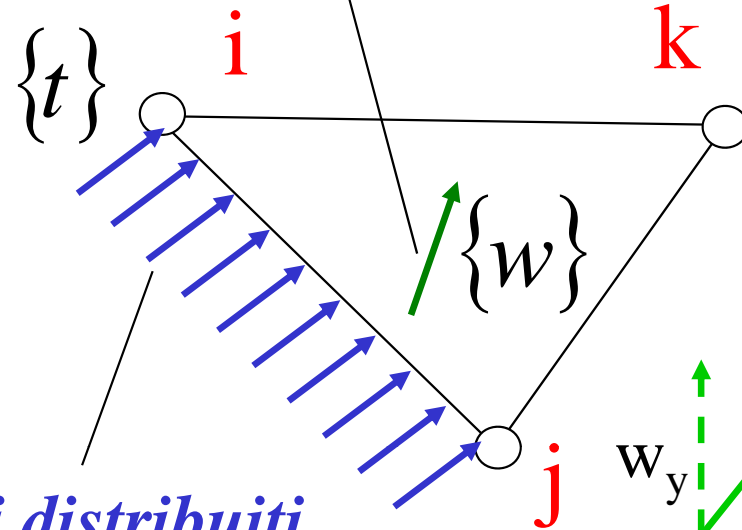
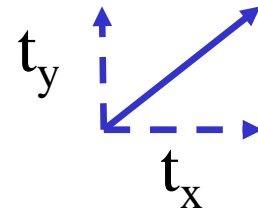
# Carichi non concentrati

*Forze di volume*

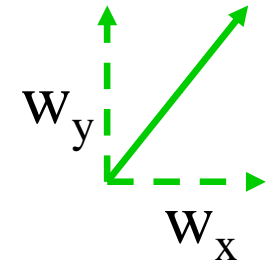
Lavoro forze di volume

$$L_{est} = \{\delta U^e\}^T \{P^e\} + L_W + L_t$$

Lavoro carichi distribuiti



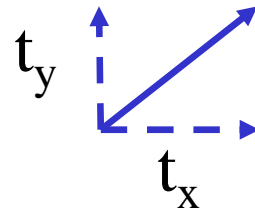
*Carichi distribuiti*



# Carichi non concentrati

*Forze di volume*

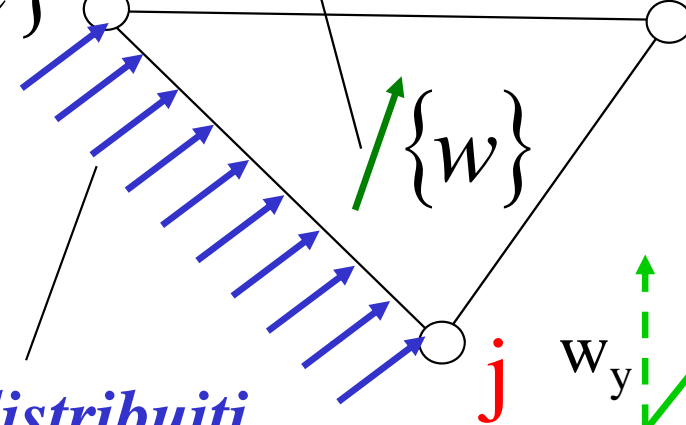
Lavoro forze di volume



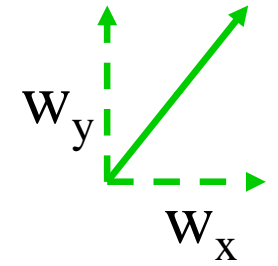
$\{t\}$

**i**

**k**



*Carichi distribuiti*



$$L_{est} = \{\delta U^e\}^T \{P^e\} + L_W + L_t$$

Lavoro carichi distribuiti

$$dL_W = \{\delta v\}^T \{w\} dV$$


$$L_W = \int_V \{\delta v\}^T \{w\} dV = \int_V \{\delta U^e\}^T [N]^T \{w\} dV = \{\delta U^e\}^T \int_V [N]^T \{w\} dV$$



$$\{P^e\} = [K^e]\{U^e\} + \{P_w^e\} + \{P_t^e\}$$





$$\{P^e\} = [K^e]\{U^e\} + \{P_w^e\} + \{P_t^e\}$$


$$\{P_w^e\} = -\int_V [N]^T \{w\} dV$$



$$\{P^e\} = [K^e]\{U^e\} + \{P_w^e\} + \{P_t^e\}$$


$$\{P_w^e\} = -\int_V [N]^T \{w\} dV$$


$$\{P_t^e\} = -\int_L [N]^T \{t\} dL$$



$$\{P^e\} = [K^e]\{U^e\} + \{P_w^e\} + \{P_t^e\}$$

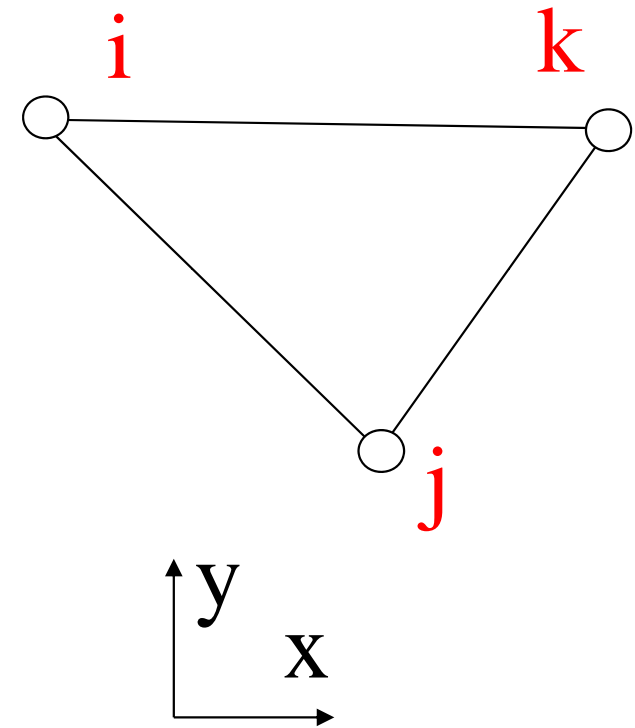
$$\{P_w^e\} = - \int_V [N]^T \{w\} dV$$

$$\{P_t^e\} = - \int_L [N]^T \{t\} dL$$

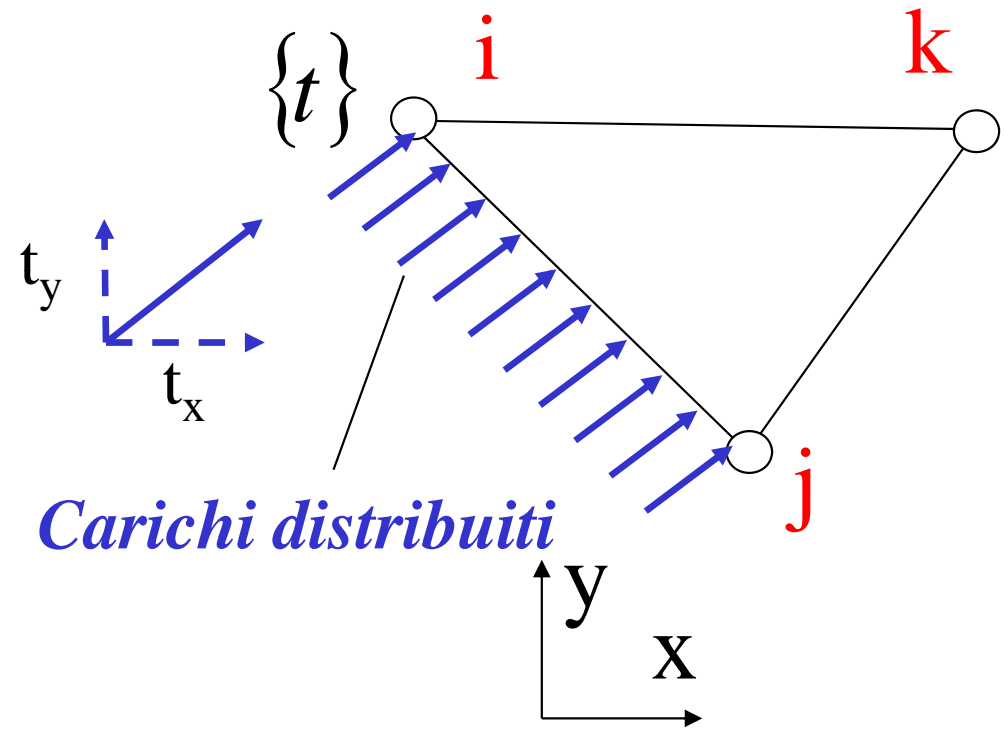
Reazioni vincolari conseguenti all'applicazione all'elemento delle forze distribuite e di volume = - Carichi nodali staticamente equivalenti alle forze distribuite o di volume



Esempio: carico uniformemente distribuito sul lato di un elemento triangolare



# Esempio: carico uniformemente distribuito sul lato di un elemento triangolare

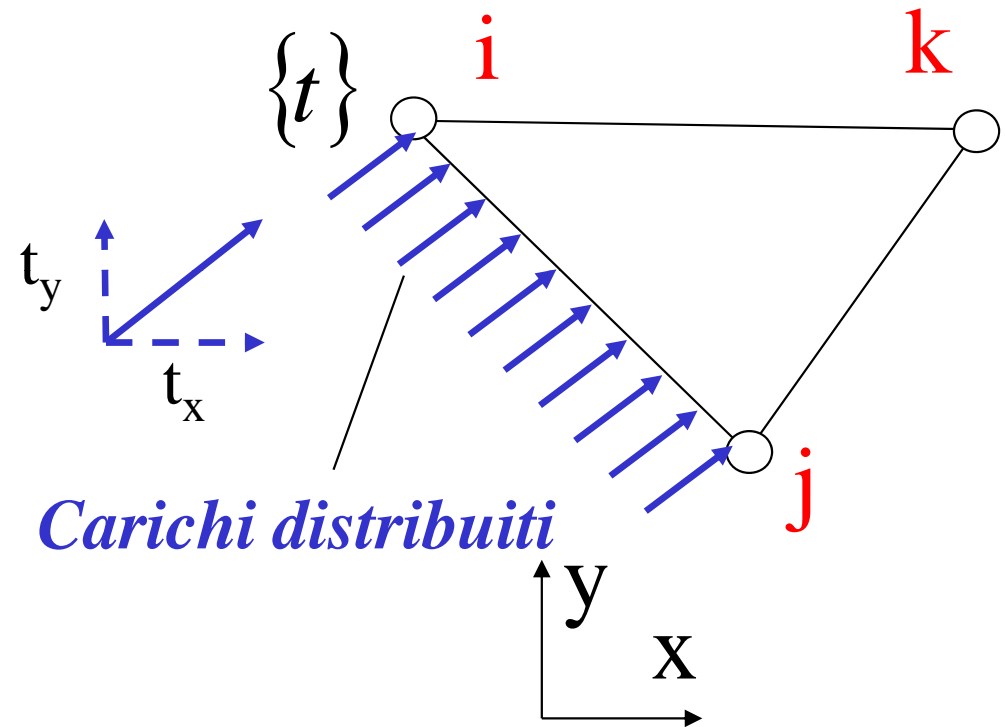




Esempio: carico uniformemente distribuito sul lato di un elemento triangolare

$$\{P_t^e\} = - \int_L [N]^T \{t\} d\xi$$

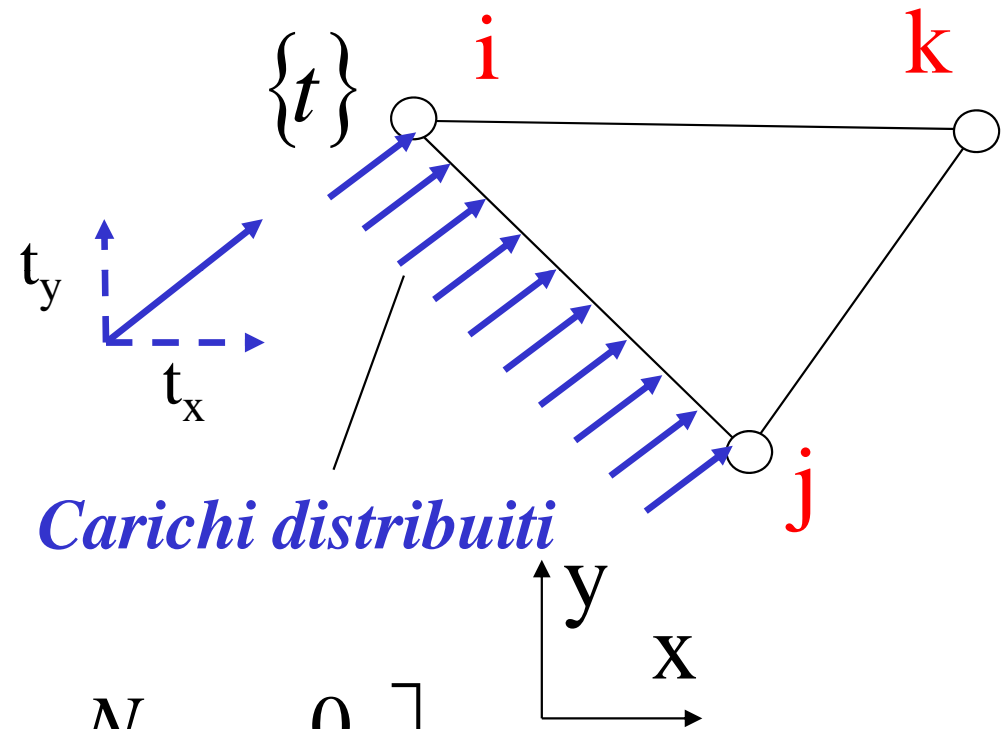
$$6 \times 1 \quad 6 \times 2 \quad 2 \times 1$$



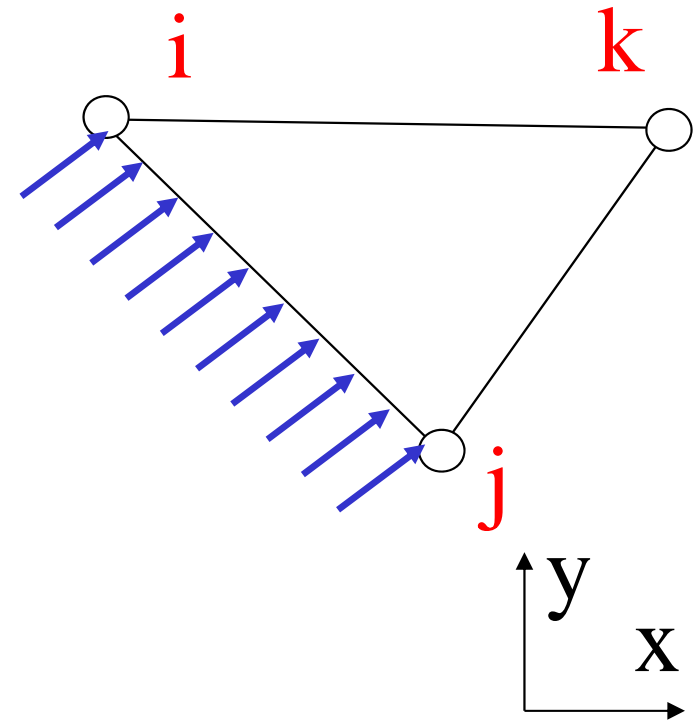
Esempio: carico uniformemente distribuito sul lato di un elemento triangolare

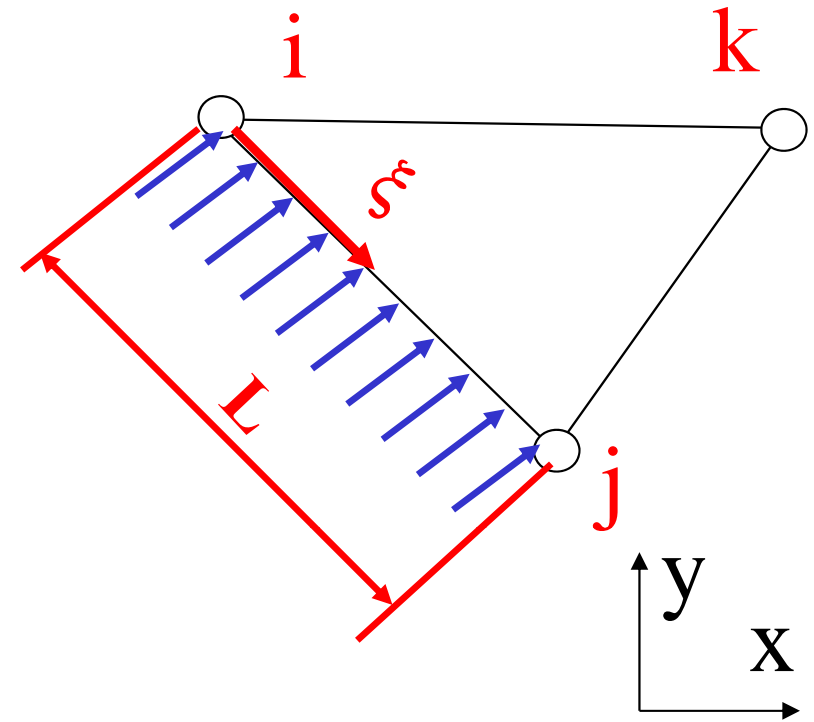
$$\{P_t^e\} = - \int_L [N]^T \{t\} d\xi$$

$$6 \times 1 \quad 6 \times 2 \quad 2 \times 1$$



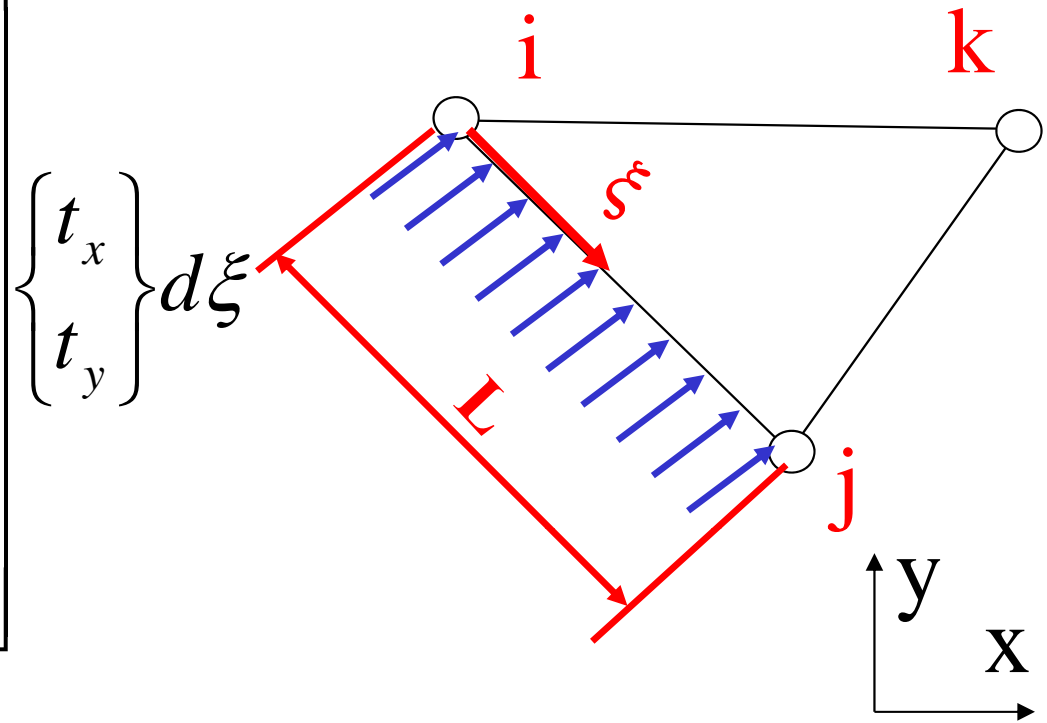
$$[N] = \begin{bmatrix} N_{11} & 0 & N_{13} & 0 & N_{15} & 0 \\ 0 & N_{11} & 0 & N_{13} & 0 & N_{15} \end{bmatrix}$$





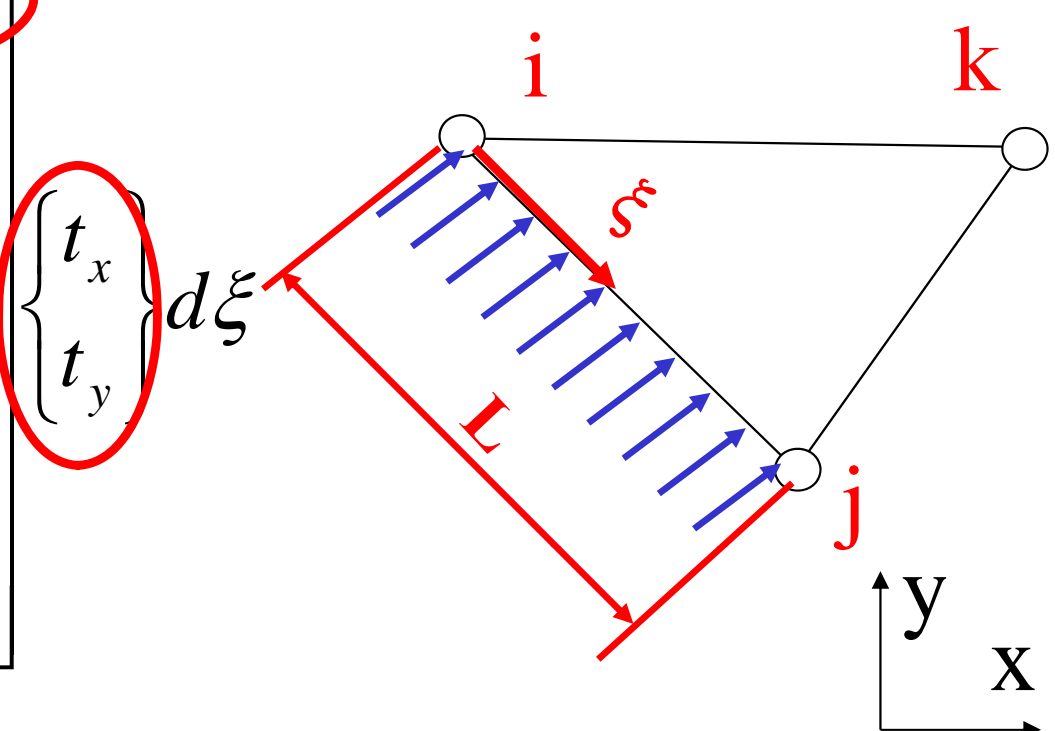


$$\{P_t^e\} = \begin{Bmatrix} p_{t,ix}^e \\ p_{t,iy}^e \\ p_{t,jx}^e \\ p_{t,jy}^e \\ p_{t,kx}^e \\ p_{t,ky}^e \end{Bmatrix} = - \int_L \begin{bmatrix} N_{11} & 0 \\ 0 & N_{11} \\ N_{13} & 0 \\ 0 & N_{13} \\ N_{15} & 0 \\ 0 & N_{15} \end{bmatrix} \begin{Bmatrix} t_x \\ t_y \end{Bmatrix} d\xi$$



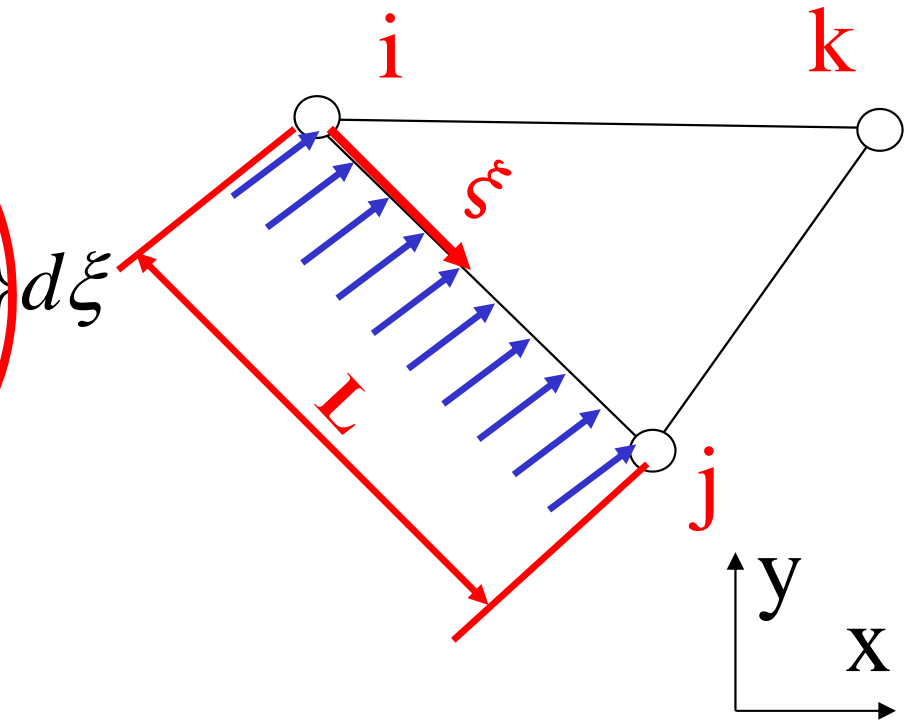


$$\{P_t^e\} = \begin{Bmatrix} p_{t,ix}^e \\ p_{t,iy}^e \\ p_{t,jx}^e \\ p_{t,jy}^e \\ p_{t,kx}^e \\ p_{t,ky}^e \end{Bmatrix} = - \int_L \begin{bmatrix} N_{11} & 0 \\ 0 & N_{11} \\ N_{13} & 0 \\ 0 & N_{13} \\ N_{15} & 0 \\ 0 & N_{15} \end{bmatrix} \begin{Bmatrix} t_x \\ t_y \end{Bmatrix} d\xi$$





$$\{P_t^e\} = \begin{Bmatrix} p_{t,ix}^e \\ p_{t,iy}^e \\ p_{t,jx}^e \\ p_{t,jy}^e \\ p_{t,kx}^e \\ p_{t,ky}^e \end{Bmatrix} = - \int_L \begin{bmatrix} N_{11} & 0 \\ 0 & N_{11} \\ N_{13} & 0 \\ 0 & N_{13} \\ N_{15} & 0 \\ 0 & N_{15} \end{bmatrix} \begin{Bmatrix} t_x \\ t_y \end{Bmatrix} d\xi$$



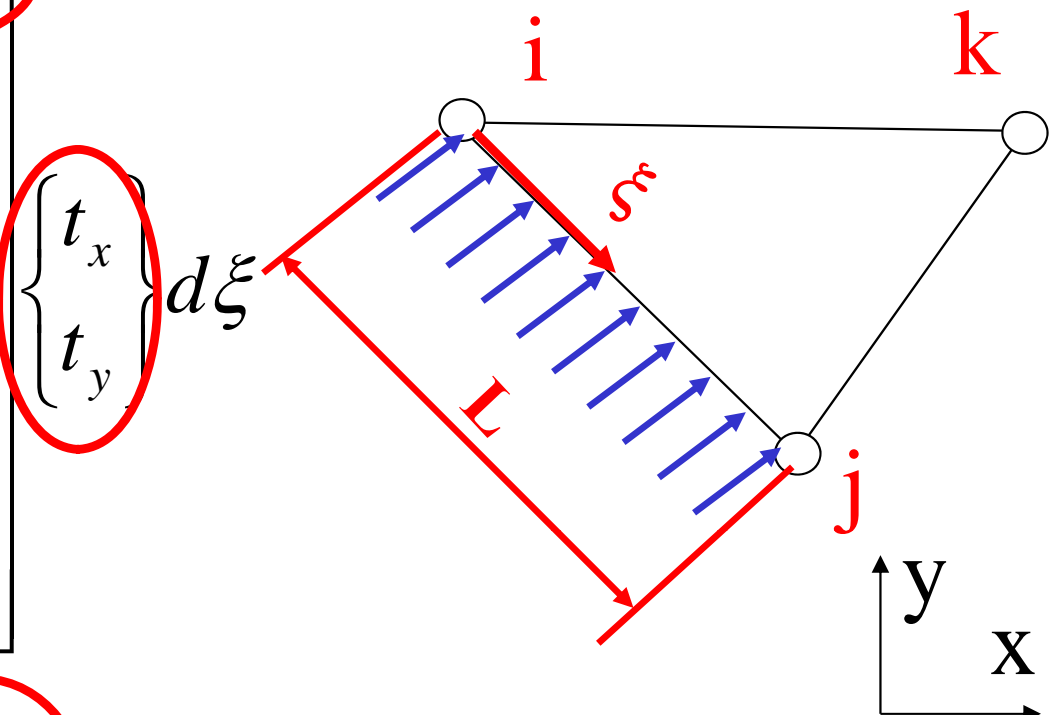
$$p_{t,ix}^e = - \int_L N_{11}(\xi) t_x d\xi = -t_x \int_L \frac{L-\xi}{L} d\xi = -\frac{t_x L}{2}$$

$$p_{t,jx}^e = - \int_L N_{13}(\xi) t_x dL = -t_x \int_L \frac{\xi}{L} dL = -\frac{t_x L}{2}$$

$$p_{t,kx}^e = - \int_L N_{15}(\xi) t_x dL = -t_x \int_L 0 dL = 0$$



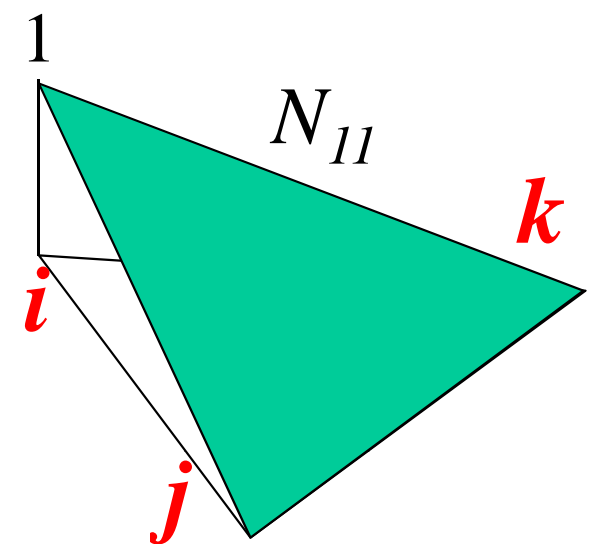
$$\{P_t^e\} = \begin{Bmatrix} p_{t,ix}^e \\ p_{t,iy}^e \\ p_{t,jx}^e \\ p_{t,jy}^e \\ p_{t,kx}^e \\ p_{t,ky}^e \end{Bmatrix} = - \int_L \begin{bmatrix} N_{11} & 0 \\ 0 & N_{11} \\ N_{13} & 0 \\ 0 & N_{13} \\ N_{15} & 0 \\ 0 & N_{15} \end{bmatrix} \begin{Bmatrix} t_x \\ t_y \end{Bmatrix} d\xi$$



$$p_{t,ix}^e = - \int_L N_{11}(\xi) t_x d\xi = -t_x \int_L \frac{L-\xi}{L} d\xi = -\frac{t_x L}{2}$$

$$p_{t,jx}^e = - \int_L N_{13}(\xi) t_x dL = -t_x \int_L \frac{\xi}{L} dL = -\frac{t_x L}{2}$$

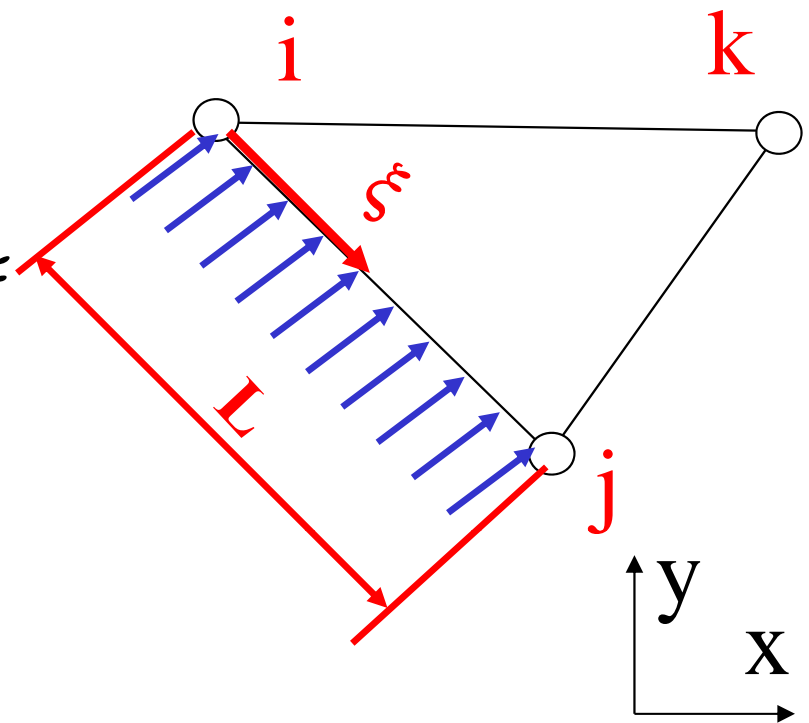
$$p_{t,kx}^e = - \int_L N_{15}(\xi) t_x dL = -t_x \int_L 0 dL = 0$$







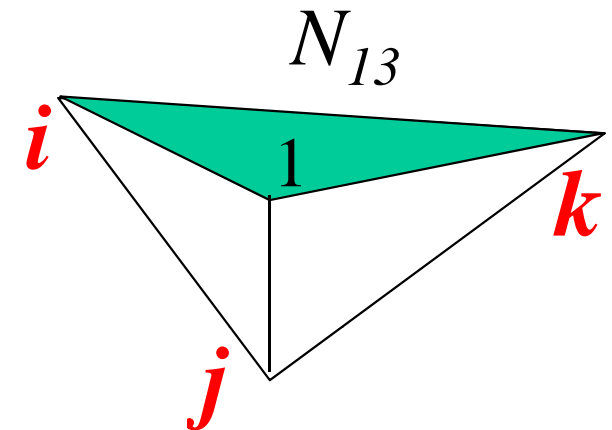
$$\{P_t^e\} = \begin{Bmatrix} p_{t,ix}^e \\ p_{t,iy}^e \\ p_{t,jx}^e \\ p_{t,jy}^e \\ p_{t,kx}^e \\ p_{t,ky}^e \end{Bmatrix} = - \int_L \begin{bmatrix} N_{11} & 0 \\ 0 & N_{11} \\ N_{13} & 0 \\ 0 & N_{13} \\ N_{15} & 0 \\ 0 & N_{15} \end{bmatrix} \begin{Bmatrix} t_x \\ t_y \end{Bmatrix} d\xi$$



$$p_{t,ix}^e = - \int_L N_{11}(\xi) t_x d\xi = -t_x \int_L \frac{L-\xi}{L} d\xi = -\frac{t_x L}{2}$$

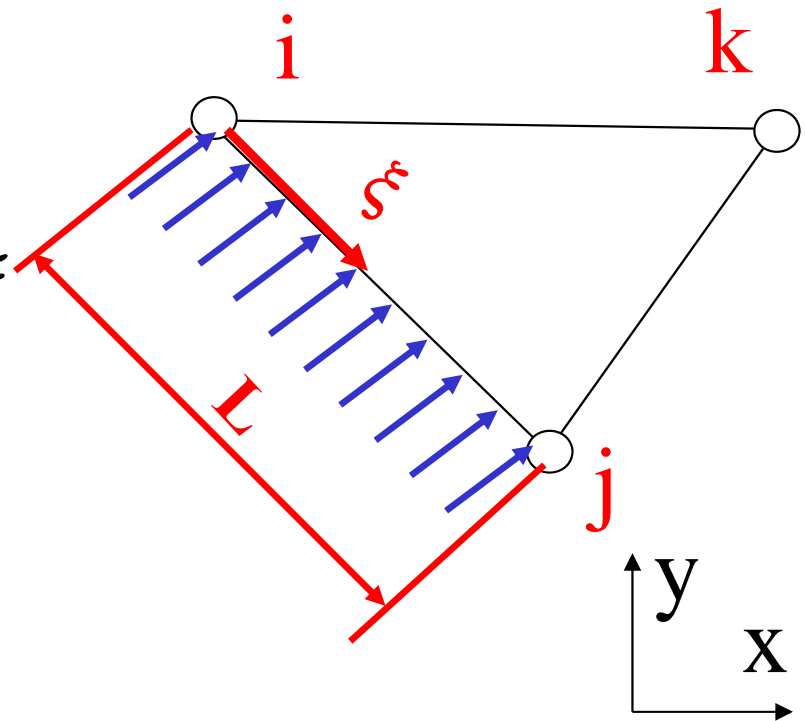
$$p_{t,jx}^e = - \int_L N_{13}(\xi) t_x dL = -t_x \int_L \frac{\xi}{L} dL = -\frac{t_x L}{2}$$

$$p_{t,kx}^e = - \int_L N_{15}(\xi) t_x dL = -t_x \int_L 0 dL = 0$$





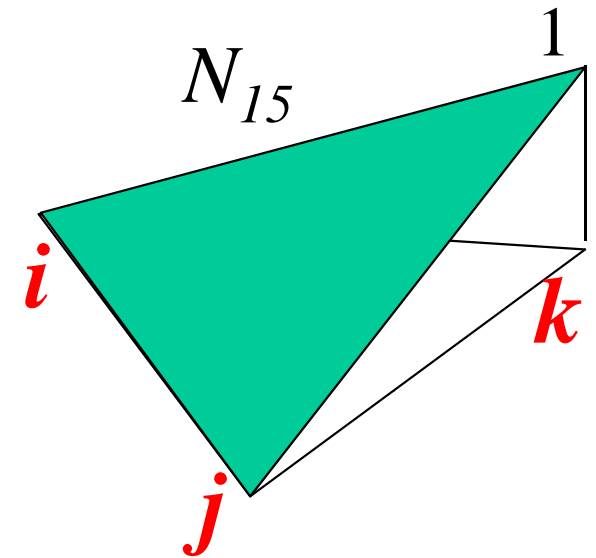
$$\{P_t^e\} = \begin{Bmatrix} p_{t,ix}^e \\ p_{t,iy}^e \\ p_{t,jx}^e \\ p_{t,jy}^e \\ p_{t,kx}^e \\ p_{t,ky}^e \end{Bmatrix} = - \int_L \begin{bmatrix} N_{11} & 0 \\ 0 & N_{11} \\ N_{13} & 0 \\ 0 & N_{13} \\ N_{15} & 0 \\ 0 & N_{15} \end{bmatrix} \begin{Bmatrix} t_x \\ t_y \end{Bmatrix} d\xi$$



$$p_{t,ix}^e = - \int_L N_{11}(\xi) t_x d\xi = -t_x \int_L \frac{L-\xi}{L} d\xi = -\frac{t_x L}{2}$$

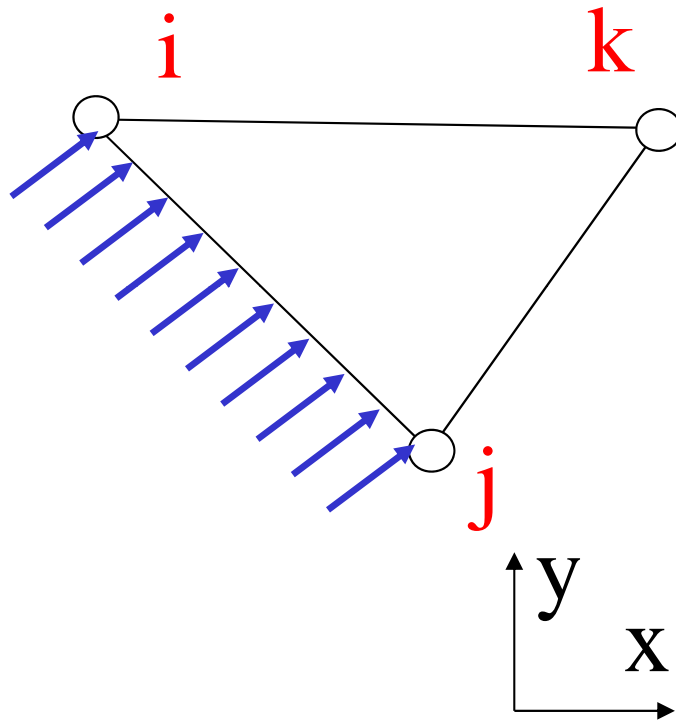
$$p_{t,jx}^e = - \int_L N_{13}(\xi) t_x dL = -t_x \int_L \frac{\xi}{L} dL = -\frac{t_x L}{2}$$

$$p_{t,kx}^e = - \int_L N_{15}(\xi) t_x dL = -t_x \int_L 0 dL = 0$$





## Carichi nodali equivalenti



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